

SINGULAR EXTENSIONS AND TRIANGULATED CATEGORIES

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Dedicated in Loving Memory of Beso Jgarkava

1. INTRODUCTION

In this paper we propose a new look on triangulated categories, which is based on singular extensions of additive categories.

Let us recall that if R is a ring and M is a square zero two-sided ideal of R , then M can be considered as a bimodule over the quotient ring $S = R/M$. Moreover the exact sequence

$$0 \rightarrow M \rightarrow R \rightarrow S \rightarrow 0$$

is a singular extension of the ring S by the bimodule M , which is characterized by an element $e(R) \in \mathrm{HH}^2(S, M)$. Here HH^* denotes the Hochschild cohomology if S is free as an abelian group and the Shukla cohomology [18], [4] in the general situation. Knowing the triple $(S, M, e(R))$ determines the ring R up to isomorphism. This classical fact admits a straightforward generalization to preadditive categories known at least from the work of Mitchell [13].

The above relates to triangulated categories as follows. Let \mathcal{T} be a triangulated category as it was introduced by Puppe [17]. Thus we do not assume the octahedron axiom of Verdier [19] to hold in \mathcal{T} . We first consider the category $\mathcal{T}^{[1]}$ of arrows of \mathcal{T} (see Section 3.1). Then for each morphism $f : A \rightarrow B$ of \mathcal{T} we choose a distinguished triangle:

$$A \xrightarrow{f} B \xrightarrow{u_f} C_f \xrightarrow{v_f} A[1].$$

Next we consider the category $\mathrm{Triangles}_0(\mathcal{T})$ which has the same objects as $\mathcal{T}^{[1]}$, while morphisms $f \rightarrow f'$ in $\mathrm{Triangles}_0(\mathcal{T})$ are commutative diagrams

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u_f} & C_f & \xrightarrow{v_f} & A[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a[1] \\ A' & \xrightarrow{f'} & B' & \xrightarrow{u_{f'}} & C_{f'} & \xrightarrow{v_{f'}} & A'[1]. \end{array}$$

Consider the functor

$$\pi : \mathrm{Triangles}_0(\mathcal{T}) \rightarrow \mathcal{T}^{[1]}$$

which is identity on objects and assigns (a, b) to the triple (a, b, c) . Obviously the functor π is identity on objects and surjective on morphisms. We prove that the kernel of the functor $\pi : \mathrm{Triangles}_0(\mathcal{T}) \rightarrow \mathcal{T}^{[1]}$ is a square zero ideal in $\mathrm{Triangles}_0(\mathcal{T})$ (see Section 3.2). It follows that there exists a bifunctor $\Theta : (\mathcal{T}^{[1]})^{\mathrm{op}} \times \mathcal{T}^{[1]} \rightarrow \mathrm{Ab}$ and a singular extension

$$0 \rightarrow \Theta \rightarrow \mathrm{Triangles}_0(\mathcal{T}) \xrightarrow{\pi} \mathcal{T}^{[1]} \rightarrow 0.$$

Hence the category $\mathrm{Triangles}_0(\mathcal{T})$ and therefore the triangulated category structure on the category \mathcal{T} is completely determined by a bifunctor Θ and the corresponding class $\vartheta \in \mathrm{HH}^2(\mathcal{T}^{[1]}, \Theta)$.

The computation of the bifunctor Θ and of the class $\vartheta \in \mathrm{HH}^2(\mathcal{T}^{[1]}, \Theta)$ is a hard problem. Of course the bifunctor Θ and the class ϑ are not arbitrary and it is an interesting task to characterize such pairs (Θ, ϑ) . In Section 6 we give a reasonable solution of this problem. Our first observation is that the categories involved in our extension possess auto-equivalences induced by the translation functor of \mathcal{T} . Thus our extension is in fact a singular τ -extension as it is defined below. Our next observation is that there exists an easily defined bifunctor Δ (called the Toda bifunctor below), which does not depend on the triangulated structure at all and is related to the bifunctor Θ via a

binatural transformation $\theta : \Delta \rightarrow \Theta$ which is an isomorphism provided one of the arguments is a split morphism of the category \mathcal{T} . Hence Δ should be considered as a first approximation of Θ . It turns out that in many cases, but not always our extension is a pushforward along θ . For example this is so if \mathcal{T} is a derived category of a ring (in the classical or in the brave new algebra sense) and it is not so if \mathcal{T} is the triangulated category constructed by Muro [14]. These facts lead to the definition of a pseudo-triangulated category in Section 4.1. We will extend the notion of homology and Massey triple product from triangulated categories to pseudo-triangulated categories. Finally in Section 6 we characterize triangulated categories among all pseudo-triangulated categories.

2. PRELIMINARIES

2.1. Pre-additive categories. A category \mathbb{A} together with an abelian group structure on each of the sets of morphisms $\text{Hom}_{\mathbb{A}}(X, Y)$ is called a *preadditive category* provided all the composition maps $\text{Hom}_{\mathbb{A}}(Y, Z) \times \text{Hom}_{\mathbb{A}}(X, Y) \rightarrow \text{Hom}_{\mathbb{A}}(X, Z)$ are bilinear maps of abelian groups. Suppose \mathbb{A} and \mathbb{B} are preadditive categories. A functor $F : \mathbb{A} \rightarrow \mathbb{B}$ is said to be an *additive functor* if the induced maps

$$\mathbb{A}(X, Y) \rightarrow \mathbb{B}(F(X), F(Y)), \quad f \mapsto F(f)$$

are homomorphisms of abelian groups for all objects $X, Y \in \mathbb{A}$.

An *additive category* is a preadditive category \mathbb{A} with zero object 0 and such that for all objects X, Y there is given an object $X \oplus Y$ and morphisms

$$\begin{aligned} i_1 : X &\rightarrow X \oplus Y, & i_2 : Y &\rightarrow X \oplus Y, \\ r_1 : X \oplus Y &\rightarrow X, & r_2 : X \oplus Y &\rightarrow Y \end{aligned}$$

with $r_1 i_1 = \text{id}_X$, $r_2 i_2 = \text{id}_Y$, $r_1 i_2 = 0$, $r_2 i_1 = 0$ and $i_1 r_1 + i_2 r_2 = \text{id}_{X \oplus Y}$. The object $X \oplus Y$ is called *direct sum* of X and Y in \mathbb{A} . It follows that $X \oplus Y$ together with i_1 and i_2 is a coproduct of X and Y and $X \oplus Y$ together with r_1 and r_2 is a product of X and Y . The following fact is well known.

Lemma 2.1.1. *For additive categories \mathbb{A} and \mathbb{B} , a functor $F : \mathbb{A} \rightarrow \mathbb{B}$ is additive iff for all objects X_1, X_2 of the category \mathbb{A} the canonical map*

$$(F(r_1), F(r_2)) : F(X_1 \oplus X_2) \rightarrow F(X_1) \oplus F(X_2)$$

is an isomorphism.

2.2. Split idempotent and split morphisms. Let $e : A \rightarrow A$ be an endomorphism. If $e^2 = e$ then e is called *idempotent*. If e is an idempotent in an additive category \mathbb{A} then $\text{id}_A - e$ is also an idempotent. For any objects X_1 and X_2 of an additive category \mathbb{A} , the morphism $e = i_1 r_1 : X_1 \oplus X_2 \rightarrow X_1 \oplus X_2$ is an idempotent. An idempotent $e : A \rightarrow A$ is called *split* if there are arrows (called *splitting data*) $a : A \rightarrow B$ and $b : B \rightarrow A$, such that $e = ba$ and $ab = \text{id}_B$. An additive category \mathbb{A} is called *Karoubian* provided all idempotents split, which is the same to require as that all idempotents have kernels (or cokernels).

A morphism $p : X \rightarrow Y$ of an additive category is called a *splittable epimorphism* if there exists a morphism $j : Y \rightarrow X$ such that $pj = \text{id}_Y$. For example the canonical projection $r : A \oplus B \rightarrow B$ is splittable. Morphisms isomorphic to such projections are called *split epimorphisms*. If \mathbb{A} is Karoubian then any splittable epimorphism is actually a split epimorphism.

Dually a morphism $i : X \rightarrow Y$ is called a *splittable monomorphism* if there exists a morphism $r : Y \rightarrow X$ such that $ri = \text{id}_X$. For example the canonical inclusion $i : A \rightarrow A \oplus B$ is splittable. Morphisms isomorphic to such inclusions are called *split monomorphisms*. If \mathbb{A} is Karoubian then any splittable monomorphism is actually a split monomorphism.

More generally a morphism $f : X \rightarrow Y$ is called *splittable* if there exist a morphism $s : Y \rightarrow X$ such that $fsf = f$. Examples of splittable morphisms are splittable epimorphisms, splittable monomorphisms and idempotents. Morphisms of the form $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : X \oplus X' \rightarrow X \oplus X''$ are split morphisms. Morphisms isomorphic to such a morphism are called *split morphisms*. If \mathbb{A} is a

Karubian category then any splittable morphism f is actually a split morphism, i. e. it can be represented as a composite ir , where r is a split epimorphism and i is a split monomorphism.

2.3. Subfunctors of additive functors and the second cross-effect. Let \mathbb{A} be an additive category. Let $F : \mathbb{A} \rightarrow \mathbf{Ab}$ be a functor with $F(0) = 0$. The *second cross-effect* functor of F is a bifunctor $\text{cr}_2(F) : \mathbb{A} \times \mathbb{A} \rightarrow \mathbf{Ab}$ defined by

$$\text{cr}_2(F)(X_1, X_2) := \text{Ker}((F(p_1), F(p_2)) : F(X_1 \oplus X_2) \rightarrow F(X_1) \oplus F(X_2)).$$

Thus a functor F is additive iff $F(0) = 0$ and $\text{cr}_2(F) = 0$.

The proof of the following fact is an easy exercise on diagram chase and is left to the reader.

Lemma 2.3.1. *For any short exact sequence of functors*

$$0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$$

one has a short exact sequence of bifunctors:

$$0 \rightarrow \text{cr}_2(F_1) \rightarrow \text{cr}_2(F) \rightarrow \text{cr}_2(F_2) \rightarrow 0$$

In particular any subfunctor of an additive functor is also additive.

□

2.4. Ideals and quotient categories. An *ideal* \mathbb{I} of \mathbb{A} is a subbifunctor of the bifunctor

$$\text{Hom}_{\mathbb{A}}(-, -) : \mathbb{A}^{\text{op}} \times \mathbb{A} \rightarrow \mathbf{Ab}.$$

It follows from Lemma 2.3.1 that \mathbb{I} is biadditive. If \mathbb{A} and \mathbb{B} are additive categories and $F : \mathbb{A} \rightarrow \mathbb{B}$ is an additive functor, one denotes by $\text{Ker}(F)$ the ideal of \mathbb{A} consisting of morphisms $f : A \rightarrow B$ such that $F(f)$ is a zero morphism in \mathbb{B} .

If \mathbb{I} is an ideal of \mathbb{A} , then one can form the quotient category \mathbb{A}/\mathbb{I} , which has the same objects as \mathbb{A} , while morphisms in \mathbb{A}/\mathbb{I} are given by

$$\text{Hom}_{\mathbb{A}/\mathbb{I}}(A, B) := \text{Hom}_{\mathbb{A}}(A, B)/\mathbb{I}(A, B).$$

One has the canonical additive functor $Q : \mathbb{A} \rightarrow \mathbb{A}/\mathbb{I}$. It is clear that $\text{Ker}(Q) = \mathbb{I}$. Any additive functor $F : \mathbb{A} \rightarrow \mathbb{B}$ factors through the category $\mathbb{A}/\text{Ker}(F)$.

2.5. Nilpotent and square zero ideals. Let \mathbb{I} and \mathbb{J} be ideals of \mathbb{A} . For all object A and B we let $\mathbb{I}\mathbb{J}(A, B)$ be the set of all products fg , where $f \in \mathbb{I}(C, B)$ and $g \in \mathbb{J}(A, C)$, for some C . We claim that $\mathbb{I}\mathbb{J}(A, B)$ is a subgroup of $\mathbb{A}(A, B)$. Indeed, if $f \in \mathbb{I}(C, B)$, $g \in \mathbb{J}(A, C)$ and $f' \in \mathbb{I}(C', B)$, $g' \in \mathbb{J}(A, C')$, then $fg + f'g' = f''g''$, where $f'' = (f, f') : C \oplus C' \rightarrow B$ and $g'' = \begin{pmatrix} g \\ g' \end{pmatrix} : A \rightarrow C \oplus C'$. We have $\mathbb{J}(A, C \oplus C') = \mathbb{J}(A, C) \oplus \mathbb{J}(A, C')$ by Lemma 2.3.1. Since $g \in \mathbb{J}(A, C)$ and $g' \in \mathbb{J}(A, C')$, it follows that $g'' \in \mathbb{J}(A, C \oplus C')$. Similarly $f'' \in \mathbb{I}(C \oplus C', B)$, hence the claim. It is clear that $\mathbb{I}\mathbb{J}$ is a subbifunctor of \mathbb{J} and \mathbb{I} . Hence it is an ideal.

Having defined product of ideals, one can talk about powers \mathbb{I}^n of an ideal \mathbb{I} . An ideal \mathbb{I} is *nilpotent* if $\mathbb{I}^n = 0$ for some n . Of special interest are ideals with $\mathbb{I}^2 = 0$. They are called *square zero* ideals. We have the following easy but useful fact.

Lemma 2.5.1. *For a square zero ideal \mathbb{I} of \mathbb{A} the bifunctor $\mathbb{I} : \mathbb{A}^{\text{op}} \times \mathbb{A} \rightarrow \mathbf{Ab}$ factors through the quotient category \mathbb{A}/\mathbb{I} in a unique way.*

□

This result can be used to prove the following simple result.

Lemma 2.5.2. *Let \mathbb{I} be a nilpotent ideal of an additive category \mathbb{A} . Then the quotient functor $Q : \mathbb{A} \rightarrow \mathbb{A}/\mathbb{I}$ reflects isomorphisms and yields an isomorphism of monoids of isomorphism classes $\text{Iso}(\mathbb{A}) \cong \text{Iso}(\mathbb{A}/\mathbb{I})$.*

Proof. The last statement follows from the previous one, because Q is identity on objects and surjective on morphisms. To prove the first statement, it suffices to assume that $\mathbb{I}^2 = 0$. Let $f : A \rightarrow B$ be a morphism, such that $Q(f)$ is an isomorphism. Thus there exists $g : B \rightarrow A$ such that $a = gf - \text{id}_A \in \mathbb{I}(A, A)$ and $b = fg - \text{id}_B \in \mathbb{I}(B, B)$. Since \mathbb{I} as a bifunctor factors through Q , it follows that the map $\mathbb{I}(B, A) \rightarrow \mathbb{I}(A, A)$ given by $x \mapsto xf$ is an isomorphism. Thus there exists a $c \in \mathbb{I}(B, A)$ with $cf = a$. Now we put $g_1 = g - c$. Then $g_1f = gf - cf = \text{id}_A$, which shows that f is a splittable monomorphism. A similar argument shows that f is a splittable epimorphism, hence an isomorphism. Thus Q reflects isomorphisms and we are done. \square

2.6. Singular extensions of additive categories. Let \mathbb{B} be an additive category and let $D : \mathbb{B}^{\text{op}} \times \mathbb{B} \rightarrow \mathbf{Ab}$ be a bifunctor. A *singular extension*

$$0 \rightarrow D \xrightarrow{i} \mathbb{A} \xrightarrow{F} \mathbb{B} \rightarrow 0$$

of \mathbb{B} by the bifunctor D is the following data:

- (i) An additive category \mathbb{A} and an additive functor $F : \mathbb{A} \rightarrow \mathbb{B}$, such that $\text{Ker}(F)$ is a square zero ideal and the canonical functor $\mathbb{A}/\text{Ker}(F) \rightarrow \mathbb{B}$ is an isomorphism of categories;
- (ii) an isomorphism of bifunctors $i : D(F(\cdot), F(\cdot)) \rightarrow \text{Ker}(F)$.

2.7. Semidirect product. Let \mathbb{B} be an additive category and let $D : \mathbb{B}^{\text{op}} \times \mathbb{B} \rightarrow \mathbf{Ab}$ be a bifunctor. The *semidirect product* (compare with [5]) of \mathbb{B} by D is the category $\mathbb{B} \ltimes D$ which has the same objects as \mathbb{B} . Morphisms $A \rightarrow B$ in $\mathbb{B} \ltimes D$ are pairs (f, a) , where $f : A \rightarrow B$ is a morphism in \mathbb{B} and $a \in D(A, B)$. Composition is defined by

$$(f, a) \circ (g, b) = (fg, f_*(b) + g^*(a))$$

Let \mathbb{I} be the class of all morphisms of the form $(0, a)$. Then $\mathbb{I}^2 = 0$, $\mathbb{A}/\mathbb{I} \cong \mathbb{B}$, where $\mathbb{A} = \mathbb{B} \ltimes D$ and $i : D \rightarrow \mathbb{I}$ is an isomorphism of bifunctors, given by $i(a) = (0, a)$. Conversely, if

$$0 \rightarrow D \xrightarrow{i} \mathbb{A} \xrightarrow{F} \mathbb{B} \rightarrow 0$$

is a singular extension and F has a section, then $\mathbb{A} \cong \mathbb{B} \ltimes D$.

2.8. Cohomology and singular extensions. The reader familiar with the Hochschild cohomology and especially with relations between the second Hochschild cohomology and singular extensions of rings might wonder whether there is a cohomology theory which in dimension two would classify singular extensions of a small additive category \mathbb{B} by a bifunctor $D : \mathbb{B}^{\text{op}} \times \mathbb{B} \rightarrow \mathbf{Ab}$. In fact such cohomology does exist and it is an obvious extension of the Shukla cohomology of rings [18], [4] to small preadditive categories.

As a matter of fact, let us mention here that there exists also Baues-Wirsching cohomology [5] which is defined for all small (maybe non-preadditive) categories. For additive categories the second Shukla cohomology and the second Baues-Wirsching cohomology $H^2(\mathbb{A}, D)$ are isomorphic. This follows from [5], together with Proposition 3.4 of [11], which shows that any linear extension [5] of an additive category by an additive bifunctor is again an additive category. It must be mentioned that even for additive categories Shukla and Baues-Wirsching cohomologies are not isomorphic in dimensions ≥ 3 .

2.9. Puppe triangulated categories. Let \mathcal{T} be an additive category with an autoequivalence $A \mapsto A[1]$. A *candidate triangle* in \mathcal{T} is a diagram

$$X \rightarrow Y \rightarrow Z \rightarrow X[1].$$

A morphism from a candidate triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ to a candidate triangle $X' \rightarrow Y' \rightarrow Z' \rightarrow X'[1]$ is a commutative diagram in \mathcal{T} :

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

We let **Cand** be the category of candidate triangles. A candidate triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ is *acyclic* provided the sequence of abelian groups

$$\cdots \rightarrow \operatorname{Hom}_{\mathcal{T}}(X[1], A) \rightarrow \operatorname{Hom}_{\mathcal{T}}(Z, A) \rightarrow \operatorname{Hom}_{\mathcal{T}}(Y, A) \rightarrow \operatorname{Hom}_{\mathcal{T}}(X, A) \rightarrow \cdots$$

is exact for any object $A \in \mathcal{T}$.

A *Puppe triangulated category* structure, or simply triangulated category structure on \mathcal{T} is given by a collection of diagrams, called *distinguished triangles*, of the form

$$X \rightarrow Y \rightarrow Z \rightarrow X[1]$$

such that

TR1) Any candidate triangle isomorphic to a distinguished triangle in **Cand** is a distinguished triangle.

TR2) Any diagram of the following form is a distinguished triangle:

$$X \xrightarrow{\operatorname{id}_X} X \rightarrow 0 \rightarrow X[1]$$

TR3) If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle, then

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is also a distinguished triangle.

TR4) For any morphism $f : X \rightarrow Y$ there is a distinguished triangle of the form

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow X[1].$$

TR5) Suppose we have a diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow a & & \downarrow & & & & \downarrow a[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

in which the rows are distinguished triangles and the left rectangle commutes. Then there exists a morphism $Z \rightarrow Z'$ making the diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow a & & \downarrow & & \downarrow & & \downarrow a[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

commute.

A category equipped with a triangulated structure is called a *triangulated category*. We let $\mathbf{Triangles}(\mathcal{T})$ be the full subcategory of **Cand** formed by distinguished triangles.

Let \mathcal{T} be a triangulated category. An additive functor $h : \mathcal{T} \rightarrow \mathbf{Ab}$ is called *homology* if, whenever

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow X[1]$$

is a distinguished triangle, the sequence

$$h(X) \rightarrow h(Y) \rightarrow h(Z) \rightarrow h(X[1])$$

is exact. Then the sequence

$$\cdots \rightarrow h^n(X) \rightarrow h^n(Y) \rightarrow h^n(Z) \rightarrow h^{n+1}(X) \rightarrow \cdots$$

is also exact, where $h^n(X) = h(X[n])$.

It is well known that the functors $\operatorname{Hom}_{\mathcal{T}}(X, -)$ and $\operatorname{Hom}_{\mathcal{T}}(-, X)$ are homologies. In particular if $X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow X[1]$ is a distinguished triangle and $h : Y \rightarrow V$ is a morphism such that $hf = 0$, then h factors through g .

3. SINGULAR EXTENSIONS AND TRIANGULATED CATEGORIES

3.1. Category of arrows. Let $[1]$ be the category associated to the ordered set $0 < 1$. For any category \mathcal{C} we let $\mathcal{C}^{[1]}$ be the category of functors $[1] \rightarrow \mathcal{C}$. Thus $\mathcal{C}^{[1]}$ is the category of arrows of \mathcal{C} . For a morphism $f : A \rightarrow B$ of the category \mathcal{C} considered as an object of the category $\mathcal{C}^{[1]}$ we use the notation \mathring{f} and the word "arrow" to denote the same morphism considered as an object of the category $\mathcal{C}^{[1]}$. Hence objects of $\mathcal{C}^{[1]}$ are arrows \mathring{f} , where $f : A \rightarrow B$ is a morphism of \mathcal{C} , while morphisms $\mathring{f} \rightarrow \mathring{f}'$ are pairs of morphisms $(a : A \rightarrow A', b : B \rightarrow B')$ in \mathcal{C} such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow a & & \downarrow b \\ A' & \xrightarrow{f'} & B' \end{array}$$

commutes.

For any object A of \mathbb{A} we write id_A for the identity morphism in \mathbb{A} and use id_A for the corresponding arrow considered as an object of $\mathcal{C}^{[1]}$. Hence $\text{id}_A = \mathring{\text{id}}_A$. Assume now that \mathcal{C} has a zero object. In this case we use the following notations. For an object A in \mathcal{C} we denote by $^A!$ (resp. $!_A$) the object of $\mathbb{A}^{[1]}$ corresponding to the unique morphism $0 \rightarrow A$ (resp. $A \rightarrow 0$) in \mathcal{C} .

The functors

$$\text{id}_?, !?, ?! : \mathcal{C} \rightarrow \mathcal{C}^{[1]}$$

are full embeddings.

3.2. The main observation. Let \mathcal{T} be a triangulated category. For each morphism $f : A \rightarrow B$ of \mathcal{T} we choose a distinguished triangle

$$(3.2.1) \quad A \xrightarrow{f} B \xrightarrow{u_f} C_f \xrightarrow{v_f} A[1],$$

where $A \mapsto A[1]$ is the translation functor. One of the axioms of triangulated categories asserts that such choice is always possible. Now we consider the category $\text{Triangles}_0(\mathcal{T})$, whose objects are morphisms of \mathcal{T} , thus the same as of the category $\mathcal{T}^{[1]}$. For a morphism $f : A \rightarrow B$ we let $[f]$ be the corresponding object of the category $\text{Triangles}_0(\mathcal{T})$. The morphisms $[f] \rightarrow [f']$ in the category $\text{Triangles}_0(\mathcal{T})$ are triples of morphisms $(a : A \rightarrow A', b : B \rightarrow B', c : C_f \rightarrow C_{f'})$ of the category \mathcal{T} such that the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u_f} & C_f & \xrightarrow{v_f} & A[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a[1] \\ A' & \xrightarrow{f'} & B' & \xrightarrow{u_{f'}} & C_{f'} & \xrightarrow{v_{f'}} & A'[1] \end{array}$$

is commutative. Thus we have full subcategories $\text{Triangles}_0(\mathcal{T}) \subset \text{Triangles}(\mathcal{T}) \subset \text{Cand}$. It is clear that the first inclusion $\text{Triangles}_0(\mathcal{T}) \subset \text{Triangles}(\mathcal{T})$ is an equivalence of categories. Moreover the category $\text{Triangles}(\mathcal{T})$ can be reconstructed from $\text{Triangles}_0(\mathcal{T})$ as follows: A candidate triangle belongs to $\text{Triangles}(\mathcal{T})$ iff if it is isomorphic (in Cand) to an object of $\text{Triangles}_0(\mathcal{T})$.

We let

$$\pi : \text{Triangles}_0(\mathcal{T}) \rightarrow \mathcal{T}^{[1]}$$

be the functor which is the identity on objects (thus $\pi([f]) = \mathring{f}$) and assigns (a, b) to the triple (a, b, c) . Another axiom of triangulated categories asserts that the functor π is surjective on morphisms.

Lemma 3.2.1. *For arbitrary object X in a triangulated category \mathcal{T} and arbitrary morphism $f : A \rightarrow B$, there exist isomorphisms*

$$\text{Hom}_{\mathcal{T}}(C_f, X) \cong \text{Hom}_{\text{Triangles}_0(\mathcal{T})}([f], !_X)$$

and

$$\text{Hom}_{\mathcal{T}}(X, C_f[-1]) \cong \text{Hom}_{\text{Triangles}_0(\mathcal{T})}(^X!, [f]).$$

These isomorphisms are natural in $X \in \mathcal{T}$ and in $f \in \text{Triangles}_0(\mathcal{T})$.

Proof. We prove the first isomorphism, second being similar. A morphism

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u_f} & C_f & \xrightarrow{v_f} & A[1] \\ \downarrow 0 & & \downarrow b & & \downarrow c & & \downarrow 0 \\ 0 & \xrightarrow{0} & X & \xrightarrow{\text{id}} & X & \xrightarrow{v_0} & 0 \end{array}$$

is uniquely determined by c , which might be arbitrary. This implies the result. \square

The following easy but extremely important fact is new.

Lemma 3.2.2. *The kernel of the functor $\pi : \text{Triangles}_0(\mathcal{T}) \rightarrow \mathcal{T}^{[1]}$ is a square zero ideal.*

Proof. Consider the following commutative diagram in \mathcal{T}

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u_f} & C_f & \xrightarrow{v_f} & A[1] \\ \downarrow 0 & & \downarrow 0 & & \downarrow c & & \downarrow 0 \\ A' & \xrightarrow{f'} & B' & \xrightarrow{u_{f'}} & C_{f'} & \xrightarrow{v_{f'}} & A'[1] \\ \downarrow 0 & & \downarrow 0 & & \downarrow c' & & \downarrow 0 \\ A'' & \xrightarrow{f''} & B'' & \xrightarrow{u_{f''}} & C_{f''} & \xrightarrow{v_{f''}} & A''[1] \end{array}$$

where rows are distinguished triangles. We have to prove that $c'c = 0$. Since $c'u_{f'} = 0$, there exist a morphism $d' : A'[1] \rightarrow C_{f''}$ such that $c' = d'v_{f'}$. Hence

$$c'c = d'v_{f'}c = d'0v_f = 0.$$

\square

Corollary 3.2.3. *There exists a well-defined bifunctor $\Theta_{\mathcal{T}}$*

$$(3.2.2) \quad \Theta_{\mathcal{T}} : (\mathcal{T}^{[1]})^{\text{op}} \times \mathcal{T}^{[1]} \rightarrow \mathbf{Ab}$$

such that

$$\Theta_{\mathcal{T}}(\mathring{f}, \mathring{f}') = \{c : C_f \rightarrow C_{f'} \mid cu_f = 0, v_{f'}c = 0\}.$$

The category $\text{Triangles}_0(\mathcal{T})$ is a singular extension of the category $\mathcal{T}^{[1]}$ by the bifunctor $\Theta_{\mathcal{T}}$,

$$(3.2.3) \quad 0 \rightarrow \Theta \rightarrow \text{Triangles}_0(\mathcal{T}) \xrightarrow{\pi} \mathcal{T}^{[1]} \rightarrow 0.$$

The class ϑ of the singular extension (3.2.2) in $\text{HH}^2(\mathcal{T}, \Theta)$ is independent of the choices of distinguished triangles (3.2.1). Hence the triangulated category structure on the category \mathcal{T} is completely determined by the bifunctor Θ and the class $\Theta_{\mathcal{T}}$.

3.3. Categories with translation. Let \mathbb{A} be an additive category. A *translation* on a category \mathbb{A} is an autoequivalence $\mathbb{A} \rightarrow \mathbb{A}$; if such a translation is fixed, then we say that \mathbb{A} is a *category with translation* or τ -category. An evaluation of the translation functor on an object A is denoted by $A[1]$ and is called translation of A . Moreover, for any object A we choose an object $A[-1]$ together with an isomorphism $(A[-1])[1] \cong A$. Then $A \mapsto A[-1]$ can be extended as a functor $(\cdot)[-1] : \mathbb{A} \rightarrow \mathbb{A}$ in a unique way. If n is an integer, then one has objects $A[n]$ defined by induction: $A[n+1] = (A[n])[1]$ if $n \geq 1$, $A[0] = A$ and $A[n-1] = (A[n])[-1]$ if $n \leq -1$. Sometimes we write $\tau(A)$ instead of $A[1]$. Of course in this case we write $\tau^n(A)$ instead of $A[n]$ as well.

Let \mathbb{A} and \mathbb{B} be categories with translation. A *translation preserving functor*, or τ -functor is an additive functor $F : \mathbb{A} \rightarrow \mathbb{B}$ such that $F(A[1]) = (F(A))[1]$ for all A .

Let \mathbb{I} be an ideal in a τ -category \mathbb{A} . We will say \mathbb{I} is a τ -ideal if for all objects A and B the isomorphism $\mathbb{A}(A, B) \rightarrow \mathbb{A}(A[1], B[1])$, $f \mapsto f[1]$, restricts to an isomorphism $\mathbb{I}(A, B) \rightarrow \mathbb{I}(A[1], B[1])$. In this case the quotient category \mathbb{A}/\mathbb{I} carries a τ -category structure and the quotient functor $\mathbb{A} \rightarrow \mathbb{A}/\mathbb{I}$ is a τ -functor. Conversely, if $F : \mathbb{A} \rightarrow \mathbb{B}$ is a τ -functor, then $\text{Ker}(F)$ is a τ -ideal.

3.4. Koszul translation. For a morphism $f : X \rightarrow Y$ in a τ -category \mathbb{A} one puts:

$$\tau(\overset{\circ}{f}) = (-f[1] : X[1] \rightarrow Y[1]).$$

Moreover, if $f' : X' \rightarrow Y'$ is another morphism of the category \mathbb{A} and $(x : X \rightarrow X', y : Y \rightarrow Y')$ is a morphism $\overset{\circ}{f} \rightarrow \overset{\circ}{f}'$ in the category $\mathbb{A}^{[1]}$, then one puts

$$\tau(x, y) = ((x[1], [y]) : \tau(\overset{\circ}{f}) \rightarrow \tau(\overset{\circ}{f}')).$$

In this way one gets a translation $\tau : \mathbb{A}^{[1]} \rightarrow \mathbb{A}^{[1]}$ called the *Koszul translation*.

Let \mathcal{T} be a triangulated category. Then $\text{Triangles}_0(\mathcal{T})$ also possesses a Koszul translation, which on objects is given by the same rule

$$\tau([f]) = (-f[1] : X[1] \rightarrow Y[1]),$$

while on morphisms it is given by $\tau(x, y, z) = (x[1], y[1], z[1])$. Here $(x[1], y[1], z[1])$ is the following morphism in $\text{Triangles}_0(\mathcal{T})$:

$$\begin{array}{ccccccc} X[1] & \xrightarrow{-f[1]} & Y[1] & \xrightarrow{-u_f[1]} & C_f[1] & \xrightarrow{-v_f[1]} & X[2] \\ \downarrow x[1] & & \downarrow y[1] & & \downarrow z[1] & & \downarrow x[2] \\ X'[1] & \xrightarrow{-f'[1]} & Y'[1] & \xrightarrow{-u_{f'}[1]} & C_{f'}[1] & \xrightarrow{-v_{f'}[1]} & A'[2] \end{array}$$

It is clear that $\pi : \text{Triangles}_0(\mathcal{T}) \rightarrow \mathcal{T}^{[1]}$ is a τ -functor.

3.5. τ -bifunctors and singular τ -extensions. Let \mathbb{A} be a τ -category. A τ -bifunctor on \mathbb{A} is a bifunctor $D : \mathbb{A}^{\text{op}} \times \mathbb{A} \rightarrow \text{Ab}$ together with a system of isomorphisms

$$t_{A,B} : D(A, B) \rightarrow D(A[1], B[1]), \quad A, B \in \mathbb{A},$$

which are natural in A and B . If D and D' are two τ -bifunctors, then a natural transformation $\xi : D \rightarrow D'$ of bifunctors is called a τ -transformation provided the following diagram commutes:

$$\begin{array}{ccc} D(A, B) & \xrightarrow{t_{A,B}} & D(A[1], B[1]) \\ \xi(A,B) \downarrow & & \downarrow \xi(A[1], B[1]) \\ D'(A, B) & \xrightarrow{t'_{A,B}} & D'(A[1], B[1]) \end{array}$$

For example the bifunctor $\text{Hom}_{\mathbb{A}}(\cdot, \cdot)$ is a τ -bifunctor, where $t_{A,B}(f) = f[1]$. Moreover, if \mathbb{I} is a τ -ideal, then it is a τ -subbifunctor of $\text{Hom}_{\mathbb{A}}$.

A singular extension

$$0 \rightarrow D \xrightarrow{i} \mathbb{B} \xrightarrow{p} \mathbb{A} \rightarrow 0$$

of a τ -category \mathbb{A} by a τ -bifunctor D is called a *singular τ -extension* if p is a τ -functor and i yields an isomorphism $D \rightarrow \text{Ker}(p)$ of τ -bifunctors over \mathbb{A} .

One easily sees that the singular extension 3.2.3 is in fact a singular τ -extension, where $\mathcal{T}^{[1]}$ and $\text{Triangles}_0(\mathcal{T})$ are equipped with Koszul translations. Here a τ -bifunctor structure on Θ , i. e. isomorphisms

$$t_{\overset{\circ}{f}, \overset{\circ}{f}'} : \Theta(\overset{\circ}{f}, \overset{\circ}{f}') \rightarrow \Theta(-\overset{\circ}{f}[1], -\overset{\circ}{f}'[1])$$

are induced by $c \mapsto c[1]$, for any $c : C_f \rightarrow C_{f'}$ with $cu_f = 0 = v_{f'}c$.

3.6. Toda bifunctor. Let \mathbb{A} be a category with translation. For morphisms $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ we consider the homomorphism of abelian groups

$$\phi_{f,f'} : \text{Hom}_{\mathbb{A}}(A[1], A') \oplus \text{Hom}_{\mathbb{A}}(B[1], B') \rightarrow \text{Hom}_{\mathbb{A}}(A[1], B')$$

given by

$$\phi_{f,f'}(g, h) = f'_*(g) - (f[1])^*(h) = f' \circ g - h \circ (f[1]).$$

Here $g : A[1] \rightarrow A'$ and $h : B[1] \rightarrow B'$ are morphisms of \mathbb{A} .

The *Toda bifunctor* $\Delta_{\mathbb{A}}$, or simply Δ is a bifunctor

$$\Delta : (\mathbb{A}^{[1]})^{\text{op}} \times \mathbb{A}^{[1]} \rightarrow \text{Ab}$$

given by

$$\Delta(\overset{\circ}{f}, \overset{\circ}{f}') := \text{Coker}(\phi_{f, f'}) = \frac{\text{Hom}_{\mathbb{A}}(A[1], B')}{f'_* \text{Hom}_{\mathbb{A}}(A[1], A') - f^* \text{Hom}_{\mathbb{A}}(B[1], B')},$$

where $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ are morphisms in \mathbb{A} .

The following lemma is straightforward.

Lemma 3.6.1. *Let \mathbb{A} be a category with translation. For any object $X \in \mathcal{C}$ and any morphism $f : A \rightarrow B$ one has*

$$\begin{aligned} \Delta(\text{Id}_X, \overset{\circ}{f}) &= 0, \\ \Delta(\overset{\circ}{f}, \text{Id}_X) &= 0, \\ \Delta(!_X, \overset{\circ}{f}) &= 0, \\ \Delta(\overset{\circ}{f}, {}^X!) &= 0, \\ \Delta({}^X!, \overset{\circ}{f}) &= \text{Coker}(\text{Hom}_{\mathbb{A}}(X[1], A) \xrightarrow{f_*} \text{Hom}_{\mathbb{A}}(X[1], B)), \\ \Delta(\overset{\circ}{f}, !_X) &= \text{Coker}(\text{Hom}_{\mathbb{A}}(B[1], X) \xrightarrow{-f^*} \text{Hom}_{\mathbb{A}}(A[1], X)). \end{aligned}$$

There is a τ -bifunctor structure on Δ which we will use throughout. The isomorphisms

$$t_{\overset{\circ}{f}, \overset{\circ}{f}'} : \Delta(\overset{\circ}{f}, \overset{\circ}{f}') \rightarrow \Delta(-\overset{\circ}{f}[1], -\overset{\circ}{f}'[1])$$

are induced by $a \mapsto a[1]$.

3.7. Natural transformation θ . Let \mathcal{T} be a triangulated category. Then we have two τ -bifunctors

$$\Theta_{\mathcal{T}}, \Delta_{\mathcal{T}} : (\mathcal{T}^{[1]})^{\text{op}} \times \mathcal{T}^{[1]} \rightarrow \mathbf{Ab}.$$

It must be noticed that the Toda bifunctor depends only on the translation structure, while the bifunctor Θ depends on the choice of the class of distinguished triangles. We now define the τ -transformation

$$\theta_{\mathcal{T}} : \Delta_{\mathcal{T}} \rightarrow \Theta_{\mathcal{T}}$$

as follows.

Let $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ be morphisms in \mathcal{T} . For any morphism $x : A[1] \rightarrow B'$ we have the following morphism of distinguished triangles:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u_f} & C_f & \xrightarrow{v_f} & A[1] \\ \downarrow 0 & & \downarrow 0 & & \downarrow c_x & & \downarrow 0 \\ A' & \xrightarrow{f'} & B' & \xrightarrow{u_{f'}} & C_{f'} & \xrightarrow{v_{f'}} & A'[1], \end{array}$$

where $c_x = u_{f'} x v_f$. One easily sees that the assignment $x \mapsto (0, 0, c_x)$ yields the homomorphism $\theta(\overset{\circ}{f}, \overset{\circ}{f}') : \Delta(\overset{\circ}{f}, \overset{\circ}{f}') \rightarrow \Theta(\overset{\circ}{f}, \overset{\circ}{f}')$, hence a natural transformation $\theta : \Delta \rightarrow \Theta$.

Proposition 3.7.1. *Let $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ be morphisms of \mathcal{T} . If f or f' is splittable, then $\theta(\overset{\circ}{f}, \overset{\circ}{f}') : \Delta(\overset{\circ}{f}, \overset{\circ}{f}') \rightarrow \Theta(\overset{\circ}{f}, \overset{\circ}{f}')$ is an isomorphism.*

Proof. It is well known that for any triangulated category \mathcal{T} the Karoubian completion \mathcal{T}^{Ka} has also a triangulated category structure and the inclusion functor i is a morphism of triangulated categories. Hence without loss of generality we may assume that all idempotents split in \mathcal{T} . By duality it suffices to consider the case, when f is splittable. Since $\mathcal{T}^{[1]}$ is an additive category, θ is a transformation of additive bifunctors and any splittable morphism considered as an object of $\mathcal{T}^{[1]}$ is isomorphic to a direct sum of objects of the form Id_A , $!_B$ or ${}^C!$, we have to consider three cases $\overset{\circ}{f} = \text{Id}_X$, $\overset{\circ}{f} = !_X$ and $\overset{\circ}{f} = {}^X!$. In the first case we have $C_f = 0$ and therefore both groups

$\Delta(\mathring{f}, \mathring{f}')$ and $\Theta(\mathring{f}, \mathring{f}')$ are trivial. If $\mathring{f} = !_X$, then we have already shown that $\Delta(\mathring{f}, \mathring{f}') = 0$. On the other hands if

$$\begin{array}{ccccccc} 0 & \xrightarrow{0} & X & \xrightarrow{\text{id}} & X & \xrightarrow{0} & 0 \\ \downarrow 0 & & \downarrow 0 & & \downarrow c & & \downarrow 0 \\ A' & \xrightarrow{f'} & B' & \xrightarrow{u_{f'}} & C_{f'} & \xrightarrow{v_{f'}} & A'[1] \end{array}$$

is a morphism in $\text{Triangles}_0(\mathcal{T})$ then $c = 0$. Hence $\Theta(\mathring{f}, \mathring{f}') = 0$ as well. Now consider the case, when $\mathring{f} = X!$. Let $c : X[1] \rightarrow C_{f'}$ be a morphism in \mathcal{T} , then

$$\begin{array}{ccccccc} X & \xrightarrow{0} & 0 & \xrightarrow{0} & X[1] & \xrightarrow{-\text{id}} & X[1] \\ \downarrow 0 & & \downarrow 0 & & \downarrow c & & \downarrow 0 \\ A' & \xrightarrow{f'} & B' & \xrightarrow{u_{f'}} & C_{f'} & \xrightarrow{v_{f'}} & A'[1] \end{array}$$

is a morphism of distinguished triangles iff

$$c \in \text{Ker}(\text{Hom}_{\mathbb{A}}(X[1], C_{f'}) \rightarrow \text{Hom}_{\mathbb{A}}(X[1], A'[1])).$$

But the last group is isomorphic to $\text{Coker}(\text{Hom}_{\mathbb{A}}(X[1], A') \rightarrow \text{Hom}_{\mathbb{A}}(X[1], B')) = \Delta(\mathring{f}, \mathring{f}')$ and we are done. \square

4. PSEUDO-TRIANGULATED CATEGORIES

4.1. Definition and examples. Let \mathcal{P} be an additive category with translation $A \mapsto A[1]$.

Definition 4.1.1. We will say that there is given a *pseudo-triangulated* category structure on \mathcal{P} if there is given a singular τ -extension

$$0 \rightarrow \mathcal{Y} \xrightarrow{i} \text{Ptr} \xrightarrow{p} \mathcal{P}^{[1]} \rightarrow 0$$

of $\mathcal{P}^{[1]}$ by a τ -bifunctor

$$\mathcal{Y} : (\mathcal{P}^{[1]})^{\text{op}} \times \mathcal{P}^{[1]} \rightarrow \text{Ab}$$

together with a τ -transformation $\varphi : \Delta \rightarrow \mathcal{Y}$ from the Toda bifunctor to \mathcal{Y} such that $\varphi(\mathring{f}, \mathring{f}') : \Delta(\mathring{f}, \mathring{f}') \rightarrow \mathcal{Y}(\mathring{f}, \mathring{f}')$ is an isomorphism provided f or f' is splittable. If additionally φ is isomorphic then we say that \mathcal{P} is equipped with a *Toda pseudo-triangulated category* structure.

Abusing notation we will say that \mathcal{P} is a pseudo-triangulated category provided such a structure is given.

We have already seen that if $\mathcal{P} = \mathcal{T}$ is a triangulated category then the extension

$$0 \rightarrow \Theta \rightarrow \text{Triangles}_0(\mathcal{T}) \xrightarrow{\pi} \mathcal{T}^{[1]} \rightarrow 0$$

together with the transformation $\theta : \Delta \rightarrow \Theta$ gives rise to a pseudo-triangulated structure on \mathcal{T} . We refer to this example as the *pseudo-triangulated category associated* to a triangulated category \mathcal{T} .

Unlike the triangulated category structure, any τ -category can be equipped with the structure of a pseudo-triangulated category: one can take $\mathcal{Y} = \Delta$, and define Ptr to be the semidirect product of $\mathcal{P}^{[1]}$ with Δ , or one can take any other singular τ -extension of $\mathcal{P}^{[1]}$ by Δ . Thus triangulated category structures on a given category might be really different.

Let \mathcal{P} be a pseudo-triangulated category. Objects of the category Ptr are the same as of $\mathcal{P}^{[1]}$, i. e. they are still arrows, but now called *pseudo-triangles*. If a morphism f is considered as a pseudo-triangle, we use the notation $[f]$ instead of f . Assume $[f]$ and $[f']$ are pseudo-triangles. Then one has the exact sequence of abelian groups

$$(4.1.1) \quad 0 \rightarrow \mathcal{Y}(\mathring{f}, \mathring{f}') \xrightarrow{i} \text{Hom}_{\text{Ptr}}([f], [f']) \rightarrow \text{Hom}_{\mathcal{P}^{[1]}}(\mathring{f}, \mathring{f}') \rightarrow 0.$$

It follows from Lemma 3.6.1 that

$$(4.1.2) \quad \text{Hom}_{\text{Ptr}}([f], [f']) = \text{Hom}_{\mathcal{P}^{[1]}}(\mathring{f}, \mathring{f}')$$

provided one of the following equations holds: $f = \text{id}_X$, $f = !_X$, $f' = \text{id}_X$, $f' = {}^X!$, for an object $X \in \mathcal{P}$. In particular

$$\text{Hom}_{\text{Ptr}}(!_A, !_X) = \text{Hom}_{\mathcal{P}[1]}(!_A, !_X) = \text{Hom}_{\mathcal{P}}(A, X).$$

It follows that the full embedding $\mathcal{P} \rightarrow \mathcal{P}^{[1]}$ given by $X \mapsto !_X$ has a unique lifting to Ptr .

Proposition 4.1.2. *If \mathcal{P} is a pseudo-triangulated category, then for any object X and for any morphism $f : A \rightarrow B$ in \mathcal{P} one has the following exact sequences*

$$\cdots \rightarrow \text{Hom}_{\mathcal{P}}(A[n+1], X) \rightarrow \text{Hom}_{\text{Ptr}}(\tau^n([f]), !_X) \rightarrow \text{Hom}_{\mathcal{P}}(B[n], X) \xrightarrow{f^*} \text{Hom}_{\mathcal{P}}(A[n], X) \rightarrow \cdots$$

and

$$\cdots \rightarrow \text{Hom}_{\mathcal{P}}(X, A[n-1]) \xrightarrow{f_*} \text{Hom}_{\mathcal{P}}(X, B[n-1]) \rightarrow \text{Hom}_{\text{Ptr}}({}^X!, \tau^n([f])) \rightarrow \text{Hom}_{\mathcal{P}}(X, A[n]) \rightarrow \cdots$$

Proof. We prove exactness only for the first sequence. The proof for the second sequence is similar and therefore we omit it. By the exact sequence 4.1.1 we have

$$0 \rightarrow \Delta(\mathring{f}, !_X) \rightarrow \text{Hom}_{\text{Ptr}}([f], !_X) \rightarrow \text{Hom}_{\mathcal{P}[1]}(\mathring{f}, !_X) \rightarrow 0.$$

It follows from the definition of the category $\mathcal{P}^{[1]}$ that for $f : A \rightarrow B$ one has the exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{P}[1]}(\mathring{f}, !_X) \rightarrow \text{Hom}_{\mathcal{P}}(B, X) \xrightarrow{f^*} (A, X).$$

This and Lemma 3.6.1 imply exactness of the following sequence:

$$\text{Hom}_{\mathcal{P}}(B[1], X) \xrightarrow{f^*} \text{Hom}_{\mathcal{P}}(A[1], X) \rightarrow \text{Hom}_{\text{Ptr}}([f], !_X) \rightarrow \text{Hom}_{\mathcal{P}}(B, X) \xrightarrow{f^*} \text{Hom}_{\mathcal{P}}(A, X).$$

Replacing f by the translations of f we get the result. \square

4.2. Homology. We would like to introduce the notion of the homology in the setup of pseudo-triangulated categories generalizing the classical notion for triangulated categories. As in algebraic topology, a homology must satisfy the exactness and excision axioms. To introduce these axioms we need some preparations.

Lemma 4.2.1. *Let \mathcal{P} be a pseudo-triangulated category. For any objects A and B of \mathcal{P} one has a natural isomorphisms*

$$\text{Hom}_{\text{Ptr}}({}^A!, !_B) \cong \text{Hom}_{\mathcal{P}}(A[1], B)$$

Proof. Since $\text{Hom}_{\mathcal{P}[1]}({}^A!, !_B) = 0$, the result follows from the exact sequence (4.1.1) and the fact that

$$\Upsilon({}^A!, !_B) = \Delta({}^A!, !_B) = \text{Hom}_{\mathcal{P}}(A[1], B).$$

\square

In particular for any object A there is a canonical morphism

$$j_A : {}^A! \rightarrow !_A[1]$$

corresponding to $\text{id}_{A[1]}$. It follows from our construction that

$$p(j_A) = 0.$$

Since $\Delta(-, {}^A!) = 0$, it follows that $\text{Hom}_{\text{Ptr}}(-, {}^A!) = \text{Hom}_{\mathcal{P}[1]}(-, {}^A!)$. In particular for any arrow $f : A \rightarrow B$ there is a canonical morphism

$$k_f : [f] \rightarrow {}^A!$$

in Ptr corresponding to the morphism $(\text{id}_A, 0) : \mathring{f} \rightarrow {}^A!$ in $\mathcal{P}^{[1]}$. By construction it is functorial in $[f] \in \text{Ptr}$.

Let $f : X \rightarrow Y$ be a morphism in a pseudo-triangulated category \mathcal{P} . Then we have the following commutative diagram in \mathcal{P}

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & B \\ \downarrow & & \downarrow \text{id} \\ A & \xrightarrow{f} & B \end{array}$$

which gives rise to the diagram in $\mathcal{P}^{[1]}$

$$!_A \xrightarrow{(0,f)} !_B \xrightarrow{(0,\text{id})} f^\circ.$$

Since $\mathcal{T}(!_X, -) = 0$, it has the unique lift

$$!_A \xrightarrow{!_f} !_B \xrightarrow{i_f} [f]$$

to \mathbf{Ptr} . Let

$$(4.2.1) \quad j_f : [f] \rightarrow !_A[1].$$

be the composite $j_f = j_A \circ k_f$.

Gluing these sequences and applying the translation functor we obtain the sequence

$$\cdots \rightarrow !_A[n] \rightarrow !_B[n] \rightarrow \tau^n([f]) \rightarrow !_A[n+1] \rightarrow \cdots$$

which is functorial in $[f] \mathbf{Ptr}$. Similarly one gets the sequence of morphisms:

$$\cdots \rightarrow A[n]! \rightarrow B[n]! \rightarrow \tau^{n+1}([f]) \rightarrow A[n+1]! \rightarrow \cdots$$

Definition 4.2.2. A morphism $x : [f] \rightarrow [g]$ in \mathbf{Ptr} is called *excising* if the induced map

$$\text{Hom}_{\mathbf{Ptr}}(X!, [f]) \rightarrow \text{Hom}_{\mathbf{Ptr}}(X!, [g])$$

is an isomorphism for any $X \in \mathcal{P}$.

Lemma 4.2.3. For any object A the natural map

$$j_A : A! \rightarrow !_A[1]$$

is *excising*.

Proof. Since $\text{Hom}_{\mathcal{P}[1]}(X!, !_Y) = 0 = \mathcal{T}(X!, Y!)$ and $\mathcal{T}(X!, !_Y) = \text{Hom}_{\mathcal{P}}(X[1], Y)$ the result follows. \square

Now we are ready to give the following definition.

Definition 4.2.4. A *homology* on a pseudo-triangulated category \mathcal{P} with values in an abelian category \mathbf{A} is a covariant functor $\mathbf{h} : \mathbf{Ptr} \rightarrow \mathbf{A}$ satisfying the following two axioms:

(Exactness) For any morphism $f : A \rightarrow B$ of the category \mathcal{P} the sequence

$$\mathbf{h}(!_A) \xrightarrow{!_f} \mathbf{h}(!_B) \xrightarrow{i_f} \mathbf{h}([f]) \xrightarrow{j_f} \mathbf{h}(!_A[1])$$

is exact.

(Excision) If $x : [f] \rightarrow [g]$ is excising then $\mathbf{h}(x) : \mathbf{h}([f]) \rightarrow \mathbf{h}([g])$ is an isomorphism.

In presence of the Excision Axiom, the Exactness Axiom is equivalent to the assertion that for any $f : A \rightarrow B$ the sequence

$$\mathbf{h}(B^{[-1]}!) \rightarrow \mathbf{h}([f]) \rightarrow \mathbf{h}(A!) \rightarrow \mathbf{h}(B!)$$

is exact. This easily follows from Lemma 4.2.3.

For a homology \mathbf{h} we put

$$\mathbf{h}^n(A) := \mathbf{h}(!_A[n]), \quad \mathbf{h}^n([f]) := \mathbf{h}(\tau^n([f])).$$

Then we have an exact sequence

$$\cdots \rightarrow h^n(A) \rightarrow h^n(B) \rightarrow h^n([f]) \rightarrow h^{n+1}(A) \rightarrow \cdots$$

natural in $[f] \in \text{Ptr}$.

Proposition 4.2.5. *For any object $X \in \mathcal{P}$ the functor*

$$\text{Hom}_{\text{Ptr}}(X!, -) : \text{Ptr} \rightarrow \text{Ab}$$

is a homology theory.

Proof. First we have to prove exactness of the sequence

$$\text{Hom}_{\text{Ptr}}(X!, !A) \rightarrow \text{Hom}_{\text{Ptr}}(X!, !B) \rightarrow \text{Hom}_{\text{Ptr}}(X!, [f]) \rightarrow \text{Hom}_{\text{Ptr}}(X!, !_{A[1]}).$$

Since $\text{Hom}_{\mathcal{P}[1]}(X!, Y_1) = 0$ for all $Y \in \mathcal{P}$, we have

$$\text{Hom}_{\text{Ptr}}(X!, !Y) = \mathcal{T}(X!, !Y) = \Delta(X!, !Y) = \text{Hom}_{\mathcal{P}}(X[1], Y),$$

thanks to Lemma 3.6.1 and Exact Sequence (4.1.1). Now exactness follows from Proposition 4.1.2. It remains to prove that the functor $\text{Hom}_{\text{Ptr}}(X!, -)$ transforms excising morphisms to isomorphisms. But this is obvious. \square

Lemma 4.2.6. *Let \mathcal{T} be a triangulated category and $E : \mathcal{T} \rightarrow \text{Ab}$ be a homology in the classical sense. Then the functor $h : \text{Triangles}_0(\mathcal{T}) \rightarrow \text{Ab}$ defined by*

$$h([f]) := E(C_f)$$

is a homology on the pseudo-triangulated category associated to the triangulated category \mathcal{T} . In this way one gets an equivalence between the category of homologies in classical and new sense.

Proof. By our definition of the category $\text{Triangles}_0(\mathcal{T})$ the assignment $f \mapsto C_f$ can be considered as a well-defined functor $\text{Triangles}_0(\mathcal{T}) \rightarrow \mathcal{T}$. By Lemma 3.2.1 a morphism $[f] \rightarrow [g]$ is excisable iff the induced morphism $C_f \rightarrow C_g$ is an isomorphism. From these facts, the first part of the statement follows.

Assume h is a homology in the new sense. For any morphism f in \mathcal{T} the morphism $(0, u_f) : [f] \rightarrow !_{C_f}$ is excising. Hence $h([f]) = E(C_f)$, where $E : \mathcal{T} \rightarrow \text{Ab}$ is given by $E(A) := h(!A)$. It follows easily from Exactness Axiom that E is a homology in the classical sense, hence the result. \square

4.3. Massey triple product. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

be a diagram in a pseudo-triangulated category \mathcal{P} . Suppose $hg = 0$ and $gf = 0$. Then we have the following commutative diagram in \mathcal{P} :

$$\begin{array}{ccc} X & \xrightarrow{\quad} & 0 \\ f \downarrow & & \downarrow 0 \\ Y & \xrightarrow{g} & Z \\ 0 \downarrow & & \downarrow h \\ 0 & \xrightarrow{\quad} & W \end{array}$$

which can be considered as the following diagram in $\mathcal{P}^{[1]}$:

$$X! \xrightarrow{(f,0)} \dot{g} \xrightarrow{(0,g)} !W.$$

Observe that the composite morphism is zero in $\mathcal{P}^{[1]}$. Since the functor $p : \text{Ptr} \rightarrow \mathcal{P}^{[1]}$ is identity on objects and surjective on morphisms the diagram can be lifted to Ptr :

$$[X!] \xrightarrow{x} [g] \xrightarrow{w} [!W],$$

where x and w are morphisms of pseudo-triangles such that $p(x) = (f, 0)$ and $p(w) = (0, h)$. Then $p(wx) = 0$, hence

$$wx \in \mathcal{T}(X^!, !_W);$$

since $X \rightarrow 0$ and $0 \rightarrow W$ are split morphisms, the last groups can be replaced by $\Delta(X^!, !_W)$. Hence Lemma 3.6.1 implies that

$$wx \in \text{Hom}_{\mathcal{P}}(X[1], W).$$

Actually, this element depends on lifting. If one chooses x_1 and w_1 instead of x and w , then we can write $x_1 = x + a$ and $w_1 = w + b$, where

$$a \in \mathcal{T}(X^!, \hat{g}) = \Delta(X^!, \hat{g}) = \text{Coker}(\text{Hom}_{\mathcal{P}}(X[1], Y) \xrightarrow{g_*} \text{Hom}_{\mathcal{P}}(X[1], Z))$$

and

$$b \in \mathcal{T}(\hat{g}, !_W) = \Delta(\hat{g}, !_W) = \text{Coker}(\text{Hom}_{\mathcal{P}}(Z[1], W) \xrightarrow{g^*} \text{Hom}_{\mathcal{P}}(Y[1], W)).$$

It follows that $w_1 x_1 = wx + bx + wa$, therefore the class $\{h, g, f\}$ of wx in the quotient

$$\frac{\text{Hom}_{\mathcal{P}}(X[1], W)}{h_* \text{Hom}_{\mathcal{P}}(X[1], Z) + f^* \text{Hom}_{\mathcal{P}}(Y[1], W)}$$

is invariant; we call it the *Massey product*. By definition we have $wx \in \{h, g, f\}$. In the case of triangulated categories it coincides with the classical Massey product as defined in [8].

The following fact is well-known [10, Theorem 13.2].

Lemma 4.3.1. *Let*

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1].$$

be an acyclic triangle in a triangulated category. Then it is a distinguished triangle if and only if $\text{id}_{A[1]} \in \{c, b, a\}$.

4.4. K_0 for pseudo-triangulated categories. Let \mathcal{P} be a small pseudo-triangulated category. We let $K_0(\mathcal{P})$ be the abelian group generated by the symbols $[X]$ where X is an object of \mathcal{P} , modulo the relations K1-K3 below.

- K1) $[0] = 0$,
- K2) $[X] = [Y]$ provided there exists an isomorphism $f : X \rightarrow Y$ in \mathcal{P} ,
- K3) For any arrows $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ in \mathcal{P} and excising morphism $x : [f] \rightarrow [f']$ in Ptr one has $[X] + [Y'] = [X'] + [Y]$.

One easily sees that this notion generalizes the Grothendieck's original definition for triangulated categories.

5. THE CLASS ϑ AS THE FIRST OBSTRUCTION

Recent work of Muro and his coauthors [14], [15] shows that not all triangulated categories have models. It turns out that the class ϑ is the first obstruction for a triangulated category to have a model. Namely we will prove that if ϑ is not lies in the image of the canonical homomorphism $\text{HH}^2(\mathcal{T}^{[1]}, \Delta) \xrightarrow{\theta} \text{HH}^2(\mathcal{T}^{[1]}, \Theta)$ then \mathcal{T} has no models. In other words we prove that if \mathcal{T} is a triangulated category associated to a stable model category or a Frobenius category then the extension (3.2.3) is a pushforward construction along the transformation $\theta : \Delta \rightarrow \Theta$ as it is defined in Section 5.1 and hence the class $\vartheta \in \text{HH}^2(\mathcal{T}^{[1]}, \Theta)$ lies in the image of the canonical homomorphism $\text{HH}^2(\mathcal{T}^{[1]}, \Delta) \xrightarrow{\theta} \text{HH}^2(\mathcal{T}^{[1]}, \Theta)$. Actually all this is an easy consequence of the work of Baues [2], [3].

We also check that for the triangulated category constructed in [14], [15] the class ϑ does not lies in the image of the homomorphism $\text{HH}^2(\mathcal{T}^{[1]}, \Delta) \xrightarrow{\theta} \text{HH}^2(\mathcal{T}^{[1]}, \Theta)$. This give an alternative proof of the corresponding result of [14], [15].

5.1. Push-forward construction and domination. Let \mathcal{P} be a τ -category equipped with a pseudo-triangulated category structure given by a singular τ -extension

$$0 \rightarrow \mathcal{Y} \xrightarrow{i} \mathbf{Ptr} \xrightarrow{p} \mathcal{P}^{[1]} \rightarrow 0$$

and a τ -transformation $\varphi : \Delta \rightarrow \mathcal{Y}$.

Assume a τ -transformation $\xi : \mathcal{Y} \rightarrow \mathcal{Y}_1$ of τ -bifunctors is given which is an isomorphism as soon as one of the arguments is a split morphism. Consider the following category \mathbf{Ptr}_1 . The objects of \mathbf{Ptr}_1 are the same as of the categories $\mathcal{P}^{[1]}$ and \mathbf{Ptr} , i. e. they are arrows of the category \mathcal{P} . Moreover $\mathrm{Hom}_{\mathbf{Ptr}_1}([f], [g])$ is defined using the pushout diagram of abelian groups:

$$\begin{array}{ccc} \mathcal{Y}(\overset{\circ}{f}, \overset{\circ}{g}) & \longrightarrow & \mathrm{Hom}_{\mathbf{Ptr}}([f], [g]) \\ \downarrow & & \downarrow \\ \mathcal{Y}_1(\overset{\circ}{f}, \overset{\circ}{g}) & \longrightarrow & \mathrm{Hom}_{\mathbf{Ptr}_1}([f], [g]) \end{array}$$

It is easy to see that in this way one gets a singular τ -extension structure on \mathbf{Ptr}_1 , such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathbf{Ptr} & \longrightarrow & \mathcal{P}^{[1]} \longrightarrow 0 \\ & & \downarrow \xi & & \downarrow j & & \downarrow \mathrm{id} \\ 0 & \longrightarrow & \mathcal{Y}_1 & \longrightarrow & \mathbf{Ptr}_1 & \longrightarrow & \mathcal{P}^{[1]} \longrightarrow 0. \end{array}$$

Hence \mathbf{Ptr}_1 together with the τ -transformation $\xi \circ \varphi : \Delta \rightarrow \mathcal{Y}_1$ is a pseudo-triangulated category structure on \mathcal{P} , called the *pushforward construction*. In this situation we also say that the pseudo-triangulated category \mathbf{Ptr} dominates \mathbf{Ptr}_1 and write $\mathbf{Ptr}_1 \leq \mathbf{Ptr}$.

The proof of the following easy fact is left to the reader.

Lemma 5.1.1. *Massey triple product is invariant under dominations.*

5.2. Toda triangulated categories. A *Toda triangulated category* is a triangulated category \mathcal{T} such that the associated pseudo-triangulated category $\mathbf{Triangles}_0$ is dominated by a Toda pseudo-triangulated category. The following is a straightforward.

Lemma 5.2.1. *A triangulated category \mathcal{T} is a Toda triangulated category iff the corresponding class $\vartheta \in \mathrm{HH}^2(\mathcal{T}^{[1]}, \Theta)$ lies in the image of the homomorphism $\mathrm{HH}^2(\mathcal{T}^{[1]}, \Delta) \rightarrow \mathrm{HH}^2(\mathcal{T}^{[1]}, \Theta)$.*

5.3. Track categories. In the recent work [2] Baues managed to construct Verdier triangulated categories from the data which he called *triangulated track categories*. Recall that a *track category* \mathbb{B} is a 2-category all of whose 2-morphisms are invertible. Such categories appeared already in the classical work [7, Ch. V]. Thus \mathbb{B} consists of objects X, Y , etc., with 1-morphisms ξ, η and with 2-morphisms $H : \xi \Rightarrow \eta$. If $\xi, \eta : X \rightarrow Y$ are 1-morphisms and there exists a 2-morphism $H : \xi \Rightarrow \eta$ then we say that ξ and η are homotopic. The corresponding quotient category is denoted by \mathbb{B}_{\sim} , which comes with the quotient functor $Q : \mathbb{B} \rightarrow \mathbb{B}_{\sim}$. Following [2] we use additive notation for the composite of 2-morphisms. A triangulated track category is a track category with some extra data. We refer to the original paper of Baues [2] for the exact definition. Here we point out that any pointed simplicial closed model category which is “stable” (meaning that the suspension induces an auto-equivalence of the homotopy category) gives rise to a triangulated track category structure on the track category \mathbb{B} , which consists of fibrant-cofibrant objects, 1-morphisms are usual morphisms, while 2-morphisms are homotopy classes of homotopies.

5.4. Hardie category. We need the following construction due to Hadrie [9] which we learned from [3]. Let \mathbb{B} be a track category. Let \mathbb{A} be the corresponding homotopy category $\mathbb{A} = \mathbb{B}_{\sim}$. For each morphism f of \mathbb{A} we choose its representative \tilde{f} in the homotopy class of f . Hence $Q(\tilde{f}) = f$.

Objects of the *Hardie category* $\mathcal{H}(\mathbb{B})$ associated to the track category \mathbb{B} are morphisms of \mathbb{A} . An object of $\mathcal{H}(\mathbb{B})$ corresponding to a morphism f is denoted by $\{f\}$. A morphism $\{f\} \rightarrow \{g\}$ in the category $\mathcal{H}(\mathbb{B})$ corresponding to $f : A \rightarrow B$ and $g : X \rightarrow Y$ is an equivalence class of triples

(ξ, η, H) , where $\xi : A \rightarrow X$ and $\eta : B \rightarrow Y$ are 1-morphisms of the track category \mathbb{B} , while H is a 2-morphism $H : \eta \circ \tilde{f} \Rightarrow \tilde{g} \circ \xi$. Two such triples (ξ, η, H) and (ξ', η', H') are equivalent if there are 2-morphisms $G : \eta' \Rightarrow \eta$ and $K : \xi \Rightarrow \xi'$ such that

$$H' = \tilde{g}K + H + \tilde{f}G.$$

Let $\{\xi, \eta, H\}$ be the equivalence class of (ξ, η, H) . Composition in the Hardie category is given by

$$\{\xi, \eta, H\} \circ \{\xi_1, \eta_1, H_1\} = \{\xi\xi_1, \eta\eta_1, \eta H_1 + \xi H\}.$$

Lemma 5.4.1. *Let \mathbb{B} be a triangulated track category. Then there is a well-defined functor $p : \mathcal{H}(\mathbb{B}) \rightarrow \mathbb{A}^{[1]}$ which is identity on objects and on morphisms is given by*

$$p\{\xi, \eta, H\} = (Q(\xi), Q(\eta))$$

Moreover, if \mathbb{B} is a triangulated track category, then p is a part of a singular τ -extension

$$0 \rightarrow \Delta \rightarrow \mathcal{H}(\mathbb{B}) \xrightarrow{p} \mathbb{A}^{[1]} \rightarrow 0.$$

Proof. This fact modulo notation is due to Baues [3]. The extension is the same as his linear extension [3, Equation (2), page 266], which is defined for much more general track categories. The only thing to check is that for triangulated track categories D^\sharp in the notation of [3] is the Toda bifunctor. But this follows immediately from the definition of D^\sharp given in [3, Equation (2.2)] and the fact that $D(X, Y) = \text{Hom}_{\mathbb{A}}(X[1], Y)$ for triangulated track categories, see [2, Equation (2.7)]. \square

5.5. Pushforward construction in action. Let \mathbb{B} be a triangulated track category. By [2] the homotopy category $\mathbb{A} := \mathbb{B}_{\simeq}$ possesses a structure of triangulated category and therefore we have a singular τ -extension (see Extension 3.2.3):

$$0 \rightarrow \Theta \rightarrow \text{Triangles}_0(\mathbb{A}) \rightarrow \mathbb{A}^{[1]} \rightarrow 0.$$

In this section we prove the following result.

Proposition 5.5.1. *Let \mathbb{B} be a triangulated track category. Then there is a functor $T : \mathcal{H}(\mathbb{B}) \rightarrow \text{Triangles}_0(\mathbb{A})$ which makes the diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Delta & \longrightarrow & \mathcal{H}(\mathbb{B}) & \xrightarrow{p} & \mathbb{A}^{[1]} \longrightarrow 0 \\ & & \downarrow \theta & & \downarrow T & & \downarrow \text{id} \\ 0 & \longrightarrow & \Theta & \longrightarrow & \text{Triangles}_0(\mathbb{A}) & \longrightarrow & \mathbb{A}^{[1]} \longrightarrow 0 \end{array}$$

commute. In particular for the class ϑ defined via Extension (3.2.3) one has

$$\vartheta = \theta_*(\beta),$$

where $\beta \in \text{HH}^2(\mathbb{A}^{[1]}, \Delta)$ is the class of the extension constructed in Lemma 5.4.1.

Proof. In the notations of [2, Section 4] the functor T is defined by

$$T(\{f\}) = (A \xrightarrow{f} B \xrightarrow{u} C_{\tilde{f}} \xrightarrow{v} A[1])$$

where $f : A \rightarrow B$ is a morphism of \mathbb{A} and $u = Q(i_{\tilde{f}})$ and $v = Q(q_{\tilde{f}})$. \square

5.6. Alternative approach. To obtain the previous result that Extension (3.2.3) for a derived category of a differential algebra or a ring spectrum is pushforward along θ instead of triangulated track categories we could have used systems of triangulated diagram categories in the sense of Franke [6]. In fact let \mathcal{K} be a such system. In particular the categories $\mathcal{K}_{\mathbf{C}}$ are given for any (finite) poset \mathbf{C} satisfying some extra conditions. It follows from these axioms that each category $\mathcal{K}_{\mathbf{C}}$ has a canonical structure of a Verdier triangulated category. These categories should be considered as refinement of the triangulated category $\mathbb{A} = \mathcal{K}_{\underline{0}}$, which is the base of the system. Here \underline{n} denotes the poset $\{0 \leq \dots \leq n\}$. Based on the spectral sequence (32) [6, Proposition I.4.10] one can prove that there is a singular τ -extension

$$0 \rightarrow \Delta \rightarrow \mathcal{K}_{\underline{1}} \rightarrow \mathbb{A} \rightarrow 0.$$

and there is a functor $T : \mathcal{K}_1 \rightarrow \text{Triangles}_0(\mathbb{A})$ which makes the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Delta & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \mathbb{A}^{[1]} \longrightarrow 0 \\ & & \downarrow \theta & & \downarrow T & & \downarrow \text{id} \\ 0 & \longrightarrow & \Theta & \longrightarrow & \text{Triangles}_0(\mathbb{A}) & \longrightarrow & \mathbb{A}^{[1]} \longrightarrow 0 \end{array}$$

commute. The construction of the functor T is similar to one constructed in the proof of Proposition 5.5.1 and is based on the cones constructed in [6, Section 1.4.6].

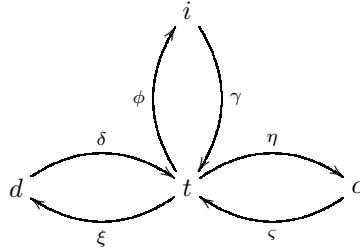
It should be point out that if a triangulated category is associated to a stable simplicial model category then both refinements – triangulated track category as well as system of triangulated diagram categories are available. One can prove that in this case Hardie category \mathcal{H} is equivalent to \mathcal{K}_1 and hence both approach gives the same singular τ -extensions.

5.7. Muro's example. For a small preadditive category S we let $\mathcal{F}(S)$ be the additive completion of S . If S has only one object (and hence S is just a ring) then $\mathcal{F}(S)$ is the category of finitely generated free S -modules. Muro [14] shoved that the category $\mathcal{F}(\mathbb{Z}/4\mathbb{Z})$ with the identity translation functor has the unique triangulated category structure such that the triangle

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z}$$

is distinguished. In this section we show that for this triangulated category the extension 3.2.3 is not a pushforward along θ . In the light of Section 5.5 and Section 5.6, it follows that this triangulated category does not admits any refinement as a triangulated track category [2] or as a system of triangulated diagram categories [6]. This fact sharpers some results from [14], [15].

Consider the preadditive category R which is generated by the following graph



modulo the following relations: all arrows are annihilated by 4 and furthermore

$$\begin{aligned} 2 \cdot \delta \xi &= 0, & 2 \cdot \varsigma \eta &= 0, \\ \eta \delta &= 0, & \phi \varsigma &= 0, & \xi \gamma &= 0, \\ \xi \delta &= 2 \cdot \text{id}_d, & \eta \varsigma &= 2 \cdot \text{id}_c, & \phi \gamma &= 2 \cdot \text{id}_i, \\ \gamma \phi &= \delta \xi + \varsigma \eta. \end{aligned}$$

Then $\text{Hom}_R(d, c) = \text{Hom}_R(c, i) = \text{Hom}_R(i, d) = \text{Hom}_R(d, c) = 0$. Moreover the abelian groups $\text{Hom}_R(d, i)$, $\text{Hom}_R(d, t)$, $\text{Hom}_R(c, d)$, $\text{Hom}_R(c, t)$, $\text{Hom}_R(i, c)$, $\text{Hom}_R(i, i)$, $\text{Hom}_R(t, d)$, $\text{Hom}_R(t, c)$ and $\text{Hom}_R(t, c)$ are isomorphic to $\mathbb{Z}/4\mathbb{Z}$. The rings $\text{Hom}_R(d, d)$, $\text{Hom}_R(c, c)$ and $\text{Hom}_R(i, i)$ are isomorphic to $\mathbb{Z}/4\mathbb{Z}$, while $\text{Hom}_R(t, t)$ as a ring is isomorphic to the ring

$$(5.7.1) \quad \text{Hom}_R(t, t) \cong \{(a, b, c) \in (\mathbb{Z}/4\mathbb{Z})^3 \mid a \equiv b \equiv c \equiv 0 \pmod{2}\}$$

This isomorphism is given by

$$(2, 2, 0) \mapsto \gamma \phi, \quad (0, 2, 2) \mapsto \delta \varsigma.$$

Let R_1 be the quotient of R by the relations

$$2\xi = 0, \quad 2\varsigma = 0, \quad \xi\varsigma = 0.$$

Finally let R_2 be the quotient of R_1 by the relation

$$\gamma \phi = 2 \cdot \text{id}_t.$$

We let $q : R \rightarrow R_2$ and $p : R_1 \rightarrow R_2$ be the quotient homomorphisms. We claim that neither q and nor p has a section. This is clear for p because even the homomorphism of abelian groups $\mathbb{Z}/4\mathbb{Z} = \text{Hom}_R(t, s) \rightarrow \text{Hom}_{R_2}(t, s) = \mathbb{Z}/2\mathbb{Z}$ does not have a section. For the functor p one observes that $p(x, y) : \text{Hom}_{R_1}(x, y) \rightarrow \text{Hom}_{R_2}(x, y)$ is an isomorphism for all possible x, y except the case when $x = y = t$. Hence, if p has a section s , then s would respects all arrows indicated in the graph. But this contradicts to the fact that the equality $\gamma\phi = 2 \cdot \text{id}_t$ holds in R_2 but not in R_1 .

Define R_2 - R_2 -bimodules Δ, Θ, Θ_1 as follows. The bifunctor Θ_1 is zero everywhere but $\Theta_1(t, t) = \mathbb{Z}/2\mathbb{Z}$. The left and right action of the endomorphism ring of t on $\Theta_1(t, t)$ is given by the multiplication on a (which is the same as the multiplication by b or c). Here we used the identification 5.7.1. Moreover, we have

$$\Delta(i, -) = 0 = \Delta(-, i), \quad \Delta(d, -) = 0 = \Delta(-, c),$$

$$\Delta(c, d) = \mathbb{Z}/4\mathbb{Z}, \quad \Delta(t, d) = \Delta(t, t) = \Delta(c, t) = \mathbb{Z}/2\mathbb{Z}.$$

The arrows of R_2 acts on Δ as follows. The homomorphisms $\Delta(c, \delta), \Delta(t, \delta)$ are natural epimorphisms $\mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$, the morphisms $\Delta(c, \xi), \Delta(\varsigma, d)$ are natural inclusions $\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z}$, finally we have $\Delta(t, \xi) = 0 = \Delta(\varsigma, t)$, while $\Delta(t, \delta), \Delta(\eta, t)$ are isomorphisms. The bifunctor Θ on objects has the same values as the bifunctor Δ and even morphisms act on Θ and Δ in the same way provided the group $\Delta(t, t)$ is not involved. The rest actions are given as follows. The morphisms $\Theta(t, \delta), \Theta(t, \xi), \Theta(\varsigma, t)$ are isomorphisms, while $\Theta(\eta, t) = 0$. Then one has a binatural transformation $\theta : \Delta \rightarrow \Theta$, such that $\theta(x, y)$ is the identity morphism for all possible x and y except the case when $x = t = y$ and in this exceptional case we have $\theta(t, t) = 0$. One observes that we have the following diagram with exact columns and rows

$$\begin{array}{ccccccc} & & \Delta & & & & \\ & & \downarrow \theta & & & & \\ 0 & \longrightarrow & \Theta & \longrightarrow & R & \xrightarrow{q} & R_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \\ 0 & \longrightarrow & \Theta_1 & \longrightarrow & R_1 & \xrightarrow{p} & R_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

We have already seen that the bottom singular extension does not split. Hence the middle singular extension is not a pushforward along θ .

All this related to Muro's example as follows. By mapping

$$d \mapsto (0 \rightarrow \mathbb{Z}/4\mathbb{Z}), \quad c \mapsto (\mathbb{Z}/4\mathbb{Z} \rightarrow 0), \quad i \mapsto (\mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/4\mathbb{Z}), \quad t \mapsto (\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z})$$

$$\phi \mapsto (2, 1), \eta \mapsto (1, 0), \delta \mapsto (0, 1), \gamma \mapsto (1, 2), \varsigma \mapsto (2, 0), \xi \mapsto (0, 2)$$

one gets an equivalence of categories:

$$\mathcal{F}(\mathbb{Z}/4\mathbb{Z})^{[1]} \cong \mathcal{F}(R_2),$$

while mapping

$$d \mapsto (0 \rightarrow \mathbb{Z}/4\mathbb{Z}), \quad c \mapsto (\mathbb{Z}/4\mathbb{Z} \rightarrow 0), \quad i \mapsto (\mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/4\mathbb{Z}), \quad t \mapsto (\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z})$$

$$\phi \mapsto (2, 1, 0), \eta \mapsto (1, 0, 2), \delta \mapsto (0, 1, 2), \gamma \mapsto (1, 2, 0), \varsigma \mapsto (2, 0, 1), \xi \mapsto (0, 2, 1)$$

one gets an equivalence of categories:

$$\text{Triangles}_0 \cong \mathcal{F}(R)$$

and these equivalences are compatible with bifunctors Δ, Θ , etc. This proves that for the Muro's triangulated category the extension 3.2.3 is not a pushforward along θ .

6. PSEUDO-TRIANGULATED VERSUS TRIANGULATED CATEGORIES

6.1. Embedding under domination. Let us recall that we have a full embedding $!_? : \mathcal{P} \rightarrow \mathcal{P}^{[1]}$. Since $\mathcal{Y}(!_X, -) = 0$ this embedding has a unique lifting $!_? : \mathcal{P} \rightarrow \mathbf{Ptr}$ which is still an embedding. Because of uniqueness it is invariant under domination. In fact we have the following result.

Lemma 6.1.1. *Let*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathbf{Ptr} & \longrightarrow & \mathcal{P}^{[1]} \longrightarrow 0 \\ & & \downarrow \xi & & \downarrow j & & \downarrow \text{id} \\ 0 & \longrightarrow & \mathcal{Y}_1 & \longrightarrow & \mathbf{Ptr}_1 & \longrightarrow & \mathcal{P}^{[1]} \longrightarrow 0 \end{array}$$

be part of a pushforward construction of pseudo-triangulated categories. Then the diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{!_?} & \mathbf{Ptr} \\ & \searrow & \downarrow j \\ & & \mathbf{Ptr}_1 \end{array}$$

commutes. Moreover the functor $!_? : \mathcal{P} \rightarrow \mathbf{Ptr}$ has a left adjoint iff the functor $!_? : \mathcal{P} \rightarrow \mathbf{Ptr}'$ does.

Proof. The first part is a consequence of the uniqueness of lifting. To prove the second part, we recall some general facts related to the adjoint functors. Let \mathcal{C} be a full subcategory of a category \mathcal{C}_1 and $x \in \mathcal{C}_1$. In these circumstances one denotes by x/\mathcal{C} the category of arrows $x \rightarrow c$, where $c \in \mathcal{C}$. It is well known that the inclusion $\mathcal{C} \subset \mathcal{C}_1$ has a left adjoint iff for all objects $x \in \mathcal{C}_1$ the category x/\mathcal{C} has an initial object.

According to Sequence 4.1.1 we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow \mathcal{Y}(\overset{\circ}{f}, !_A) & \longrightarrow & \text{Hom}_{\mathbf{Ptr}}([f], !_A) & \longrightarrow & \text{Hom}_{\mathcal{P}^{[1]}}(\overset{\circ}{f}, !_A) & \longrightarrow & 0 \\ & & \downarrow j & & \downarrow \text{id} & & \\ 0 \rightarrow \mathcal{Y}'(\overset{\circ}{f}, !_A) & \longrightarrow & \text{Hom}_{\mathbf{Ptr}'}([f], !_A) & \longrightarrow & \text{Hom}_{\mathcal{P}^{[1]}}(\overset{\circ}{f}, !_A) & \longrightarrow & 0. \end{array}$$

Since $\mathcal{Y}(-, !_A) \rightarrow \mathcal{Y}'(-, !_A)$ is an isomorphism, the middle vertical map is also an isomorphism. It follows that for a fixed f the category of arrows $[f] \rightarrow !_A$ in \mathbf{Ptr} where A runs over \mathcal{P} , and the category of arrows $[f] \rightarrow !_A$ in \mathbf{Ptr}' , $A \in \mathcal{P}$, are equivalent. From this, the result follows. \square

A similar fact is true for $?!$ as well.

6.2. The main result.

Theorem 6.2.1. *Let \mathcal{P} be a τ -category equipped with a pseudo-triangulated category structure given by a singular τ -extension*

$$0 \rightarrow \mathcal{Y} \xrightarrow{i} \mathbf{Ptr} \xrightarrow{p} \mathcal{P}^{[1]} \rightarrow 0.$$

Assume the functor $!_? : \mathcal{P} \rightarrow \mathbf{Ptr}$ has a left adjoint functor $L : \mathbf{Ptr} \rightarrow \mathcal{P}$ with counit of the adjunction $w_f : [f] \rightarrow !_L([f])$, where $f : A \rightarrow B$ is a morphism in \mathcal{P} . Declare a triangle

$$X \rightarrow Y \rightarrow Z \rightarrow X[1]$$

to be distinguished provided there is a morphism $f : A \rightarrow B$ and a commutative diagram in \mathcal{P}

$$(6.2.1) \quad \begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u_f} & L([f]) & \xrightarrow{v_f} & A[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a[1] \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

where a, b, c are isomorphisms in \mathcal{P} , $!_{u_f} = w_f i_f$, while $v_f : L([f]) \rightarrow A[1]$ is the unique morphism such that $j_f = v_f \circ w_f$. Here j_f and i_f are maps in the sequence

$$!_A \xrightarrow{!_f} !_B \xrightarrow{i_f} [f] \xrightarrow{j_f} !_{A[1]}$$

constructed in Section 4.2. With this class of distinguished triangles Axioms TR1, TR2, TR4, TR5 of triangulated categories hold.

Moreover TR3 holds (and hence \mathcal{P} is a triangulated category) iff the functor $[f] \mapsto L(f)[-1]$ is right adjoint to the functor $\mathcal{P} \rightarrow \mathbf{Ptr}$ given by $X \mapsto X!$ and additionally

$$\{v_f, u_f, f\} = \text{id}_{A[1]}$$

holds for all f . If this is so then $\text{Triangles}_0(\mathcal{P}) \leq \mathbf{Ptr}$.

Proof. The functor L has the following universal property: for any morphism $x : [f] \rightarrow !_X$ in \mathbf{Ptr} there exists a unique morphism $g : L([f]) \rightarrow X$ in \mathcal{P} such that $x = !_g \circ w_f$. Applying this to the sequence (referred as the *pretriangle* corresponding to $[f]$)

$$!_A \xrightarrow{!_f} !_B \xrightarrow{i_f} [f] \xrightarrow{j_f} !_{A[1]}$$

we see that v_f indeed exists and is unique. Since $!_?$ is full and faithful we have $L(!_A) = A$. By applying the functor L to the pretriangle we obtain the sequence

$$(6.2.2) \quad A \xrightarrow{f} B \xrightarrow{u_f} L([f]) \xrightarrow{v_f} A[1],$$

in \mathcal{P} . Now we are in a position to check Axioms. The axioms TR1 and TR4 are obvious. Now we verify the axiom TR5. Consider a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u_f} & L([f]) & \xrightarrow{v_f} & A[1] \\ \downarrow a & & \downarrow b & & & & \downarrow a[1] \\ A' & \xrightarrow{f'} & B' & \longrightarrow & L([f']) & \longrightarrow & A'[1]. \end{array}$$

Then $(a, b) : \mathring{f} \rightarrow \mathring{f}'$ is a morphism in $\mathcal{P}^{[1]}$. Hence there exists a morphism $x : [f] \rightarrow [g]$ in \mathcal{P} such that $p(x) = (a, b)$. By functoriality x induces corresponding morphism on pretriangles

$$\begin{array}{ccccccc} !_A & \xrightarrow{!_f} & !_B & \xrightarrow{i_f} & [f] & \xrightarrow{j_f} & !_{A[1]} \\ \downarrow !_a & & \downarrow !_b & & \downarrow x & & \downarrow !_{a[1]} \\ !_A' & \xrightarrow{!_{f'}} & !_B' & \longrightarrow & [f'] & \longrightarrow & !_{A'[1]} \end{array}$$

Applying L we finally get the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u_f} & L([f]) & \xrightarrow{v_f} & A[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a[1] \\ A' & \xrightarrow{f'} & B' & \longrightarrow & L([f']) & \longrightarrow & A'[1] \end{array}$$

where $c = L(x)$.

To verify the axiom TR2, we take $f = \text{id}_A$. By adjointness we have

$$\text{Hom}_{\mathbf{Ptr}}(\text{Id}_A, !_B) = \text{Hom}_{\mathcal{P}}(L(\text{Id}_A), B).$$

Since $\text{Hom}_{\mathcal{P}[1]}(\text{Id}_A, !_B) = 0$ and $\Upsilon(\text{Id}_A, -) = \Delta(\text{Id}_A, -) = 0$, it follows that $\text{Hom}_{\mathbf{Ptr}}(\text{Id}_A, !_B) = 0$ for all $B \in \mathcal{P}$. Now the Yoneda lemma shows that $L(\text{Id}_A) = 0$. It follows that

$$A \xrightarrow{\text{id}} A \rightarrow 0 \rightarrow A[1]$$

is a distinguished triangle.

If Axiom TR3 holds, then $\{v_f, u_f, f\} = \text{id}_{A[1]}$ by Lemma 4.3.1. It follows from Lemma 3.2.1 and analogue of Lemma 6.1.1 for right adjoint functors that the functor $[f] \rightarrow L(f)[-1]$ is indeed the right adjoint functor.

Conversely, assume $\{v_f, u_f, f\} = \text{id}_{A[1]}$ holds for all f and the the functor $[f] \rightarrow L(f)[-1]$ is the right adjoint to the functor $!^1$. We have to check Axiom TR3.

We start with observation that the functor L takes any excising morphism into isomorphism. In fact if $x : [f] \rightarrow [g]$ is a excising, then all morphisms

$$\text{Hom}_{\mathcal{P}}(X, L(f)[-1]) \rightarrow \text{Hom}_{\text{Ptr}}(X!, [f]) \rightarrow \text{Hom}_{\text{Ptr}}(X!, [g]) \rightarrow \text{Hom}_{\mathcal{P}}(X, L(g)[-1])$$

are isomorphism for all object $X \in \mathcal{T}$. It follows from the Yoneda lemma that $L(f) \rightarrow L(g)$ is also an isomorphism.

Next, remark that for any $f : A \rightarrow B$ there are morphisms $x : A! \rightarrow [u_f]$ and $w : [u_f] \rightarrow !_{A[1]}$ in Ptr such that $p(x) = (f, 0)$, $p(w) = (0, v_f)$ and $wx \in \Upsilon(A!, !_{A[1]}) = \text{Hom}_{\mathcal{P}}(A[1], A[1])$ represents the identity morphism $\text{id}_{A[1]}$. Hence $wx = j_A$ is an excising morphism. It follows that $L(wx)$ is an isomorphism. Thus $L(w) : L([u_f]) \rightarrow A[1]$ is a split epimorphism. By 5-Lemma applied to exact sequences induced by u_f and f it follows that $L(w)$ is in fact an isomorphism. This fact implies TR3. Hence \mathcal{P} is a triangulated category. The proof also shows that the triangulated category structure is dominated by Ptr . \square

Now we are in a position to prove our main result.

Corollary 6.2.2. *Let \mathcal{P} be a τ -category equipped with a pseudo-triangulated category structure given by a singular τ -extension*

$$0 \rightarrow \Upsilon \xrightarrow{i} \text{Ptr} \xrightarrow{p} \mathcal{P}^{[1]} \rightarrow 0.$$

Then the following conditions are equivalent

- i) *There is a triangulated category structure $\text{Triangles}(\mathcal{P})$ on \mathcal{P} and a domination*

$$\text{Triangles}_0(\mathcal{P}) \leq \text{Ptr}.$$

- ii) *There is a functor $L : \text{Ptr} \rightarrow \mathcal{P}$ which is left adjoint to the functor $X \mapsto !_X$, while $[-1] \circ L$ is a right adjoint to the functor $X \mapsto X!$ and $\{v_f, u_f, f\} = \text{id}_{A[1]}$ for all f .*

Proof. The implication i) \Rightarrow ii) follows from Lemma 3.2.1 the functor together with Lemma 6.1.1 and Lemma 4.3.1. The implication ii) \Rightarrow i) follows from Theorem . \square

7. IDEMPOTENT COMPLETION

7.1. Karoubization. Any additive category \mathbb{A} has a Karoubian completion \mathbb{A}^{Ka} , which is a Karoubian category with a full embedding $i : \mathbb{A} \rightarrow \mathbb{A}^{\text{Ka}}$ satisfying the following property. If \mathbb{B} is a Karoubian category and $j : \mathbb{A} \rightarrow \mathbb{B}$ is an additive functor, then there exists an essentially unique functor $f : \mathbb{A}^{\text{Ka}} \rightarrow \mathbb{B}$ with $j = fi$. Objects of \mathbb{A}^{Ka} are pairs (A, e) , where A is an object of \mathbb{A} and $e : A \rightarrow A$ is an idempotent. A morphism $(A, e) \rightarrow (A', e')$ is a morphism $f : A \rightarrow A'$ in \mathbb{A} such that

$$fe = e'f = f.$$

Let us observe that the identity morphism of (A, e) is e and the functor i is given by $i(A) = (A, \text{id}_A)$.

Lemma 7.1.1. *An idempotent e of the category \mathbb{A} is split iff (A, e) as an object of \mathbb{A}^{Ka} is isomorphic to an object of the image of the functor $i : \mathbb{A} \rightarrow \mathbb{A}^{\text{Ka}}$.*

Proof. One easily checks that having mutually inverse morphisms

$$a : (A, e) \rightarrow (B, \text{id}_B), \quad b : (B, \text{id}_B) \rightarrow (A, e)$$

is exactly the same as to have a splitting data for e . \square

7.2. Lifting of idempotents. It is well known that if $R \rightarrow S$ is a surjective homomorphism of rings with nilpotent kernel then any idempotent of S is an image of an idempotent of R . We can specialize this for the ring homomorphism $\text{Hom}_{\mathbb{A}}(A, A) \rightarrow \text{Hom}_{\mathbb{A}/\mathbb{I}}(A, A)$ to get the following fact.

Lemma 7.2.1. *Let \mathbb{I} be a nilpotent ideal of an additive category \mathbb{A} . For any idempotent $f : A \rightarrow A$ in the quotient category \mathbb{A}/\mathbb{I} there is an idempotent $e : A \rightarrow A$ in \mathbb{A} such that $Q(e) = f$.*

7.3. Singular extensions and idempotent completion. Let

$$0 \rightarrow D \xrightarrow{i} \mathbb{A} \xrightarrow{F} \mathbb{B} \rightarrow 0$$

be a singular extension of an additive category \mathbb{A} by a biadditive bifunctor $D : \mathbb{A}^{\text{op}} \times \mathbb{A} \rightarrow \mathbf{Ab}$. In this section we compare the categories \mathbb{A}^{Ka} and \mathbb{B}^{Ka} . Actually our results here are very particular case of much more general results obtained in [16].

To make notations simpler we write $\bar{f} : A \rightarrow B$ instead of $F(f) : A \rightarrow B$. Here $f : A \rightarrow B$ is a morphism in \mathbb{A} . Since all idempotents in \mathbf{Ab} splits the bifunctor D has the canonical extension $D^{\text{Ka}} : (\mathbb{A}^{\text{Ka}})^{\text{op}} \times \mathbb{A}^{\text{Ka}} \rightarrow \mathbf{Ab}$ to the category \mathbb{A}^{Ka} . In more details

$$D^{\text{Ka}}((A, \bar{e}), (A', \bar{e}')) = \text{Im}(\bar{e}'_* \circ \bar{e}^* : D(A, A') \rightarrow D(A, A')).$$

This works because any idempotent in \mathbb{B} has the form \bar{e} for an idempotent e in \mathbb{A} thanks to Lemma 7.2.1. Now we fix idempotents $e : A \rightarrow A$, $e' : A' \rightarrow A'$. Since

$$\text{Hom}_{\mathbb{A}^{\text{Ka}}}((A, e), (A', e')) = \text{Im}(e'_* \circ e^* : \text{Hom}_{\mathbb{A}}(A, A') \rightarrow \text{Hom}_{\mathbb{A}}(A, A'))$$

the inclusion $i : D(A, A') \rightarrow \text{Hom}_{\mathbb{A}}(A, A')$ has the unique extension

$$i^{\text{Ka}} : D^{\text{Ka}}((A, \bar{e}), (A', \bar{e}')) \rightarrow \text{Hom}_{\mathbb{A}^{\text{Ka}}}((A, e), (A', e')).$$

This allows to consider D^{Ka} as a square zero ideal in \mathbb{A}^{Ka} . The corresponding quotient category is denoted by $\tilde{\mathbb{B}}$. The objects of the category $\tilde{\mathbb{B}}$ are pairs (A, e) , where e is an idempotent in the category \mathbb{A} . The morphisms are

$$\text{Hom}_{\tilde{\mathbb{B}}}((A, e), (A', e')) = \text{Im}(\bar{e}'_* \circ \bar{e}^* : \text{Hom}_{\mathbb{B}}(A, A') \rightarrow \text{Hom}_{\mathbb{B}}(A, A')).$$

This follows from the fact that the composites $e'_* \circ e^*$ and $\bar{e}'_* \circ \bar{e}^*$ are idempotents and hence their images are in fact direct summands. It follows that the functor $\tilde{B} \rightarrow \mathbb{B}^{\text{Ka}}$ defined on objects by $(A, e) \mapsto (A, \bar{e})$ is full and faithful and in fact an equivalence thanks to Lemma 7.2.1. Having this equivalence in mind the bifunctor D^{Ka} can be considered as a bifunctor on \tilde{B} . We can now summarize our discussion.

Lemma 7.3.1. *If*

$$0 \rightarrow D \xrightarrow{i} \mathbb{A} \xrightarrow{F} \mathbb{B} \rightarrow 0$$

is a singular extension of additive categories, then we have a singular extension of additive categories

$$0 \rightarrow D^{\text{Ka}} \xrightarrow{i} \mathbb{A}^{\text{Ka}} \xrightarrow{\bar{F}} \tilde{\mathbb{B}} \rightarrow 0$$

and an equivalence of categories $\tilde{\mathbb{B}} \rightarrow \mathbb{B}^{\text{Ka}}$. Moreover, we have also a singular extension of additive categories

$$0 \rightarrow D^{\text{Ka}} \rightarrow \tilde{A} \rightarrow \mathbb{B}^{\text{Ka}} \rightarrow 0$$

and an equivalence of categories

$$\tilde{A} \rightarrow \mathbb{A}^{\text{Ka}}.$$

Proof. Only the second part of the statements needs some comments. It follows from the first part by notice that an equivalence of categories yields an isomorphism in the Baues-Wirsching cohomology [5]. □

Lemma 7.3.1 says that up to equivalence of categories a singular extension gives rise to a singular extension by passing through the idempotent completion. Based on this fact we now prove the following easy fact.

Proposition 7.3.2. *Let*

$$0 \rightarrow D \xrightarrow{i} \mathbb{A} \xrightarrow{F} \mathbb{B} \rightarrow 0$$

be a singular extensions of additive categories and let $e : A \rightarrow A$ be an idempotent in \mathbb{A} . Then e splits iff $F(e)$ splits.

Proof. Of course any functor takes split idempotents to split ones. Assume now $\bar{e} = F(e)$ is split. This means that there is an isomorphism $(A, \bar{e}) \rightarrow (B, \text{id}_B)$ in \mathbb{B}^{Ka} (see Lemma 7.1.1). But both (A, \bar{e}) and (B, id_B) are in the image of the functor $\mathbb{B} \rightarrow \mathbb{B}^{\text{Ka}}$, which is an equivalence of categories (see Lemma 7.3.1). Hence there is an isomorphism $\tilde{x} : (A, e) \rightarrow (B, \text{id})$ in \mathbb{B} . The functor $\tilde{F} : \mathbb{A}^{\text{Ka}} \rightarrow \mathbb{B}$ is full and reflects isomorphisms thanks to Lemma 7.3.1 and Lemma 2.5.2. It follows that there is an isomorphism $x : (A, e) \rightarrow (B, \text{id}_B)$ and the result follows. \square

7.4. Split idempotents in the category of arrows. Let $f : A \rightarrow B$ be a morphism of an additive category \mathbb{A} . It is clear that a morphism $(a, b) : \mathring{f} \rightarrow \mathring{f}$ in the category $\mathbb{A}^{[1]}$ is an idempotent iff $a : A \rightarrow A$ and $b : B \rightarrow B$ are idempotents.

Lemma 7.4.1. *An idempotent $(a, b) : \mathring{f} \rightarrow \mathring{f}$ of $\mathbb{A}^{[1]}$ splits iff a and b are split idempotents of the category \mathbb{A} .*

Proof. Assume a and b are split idempotents of the category \mathbb{A} . Let $A \xrightarrow{c} C \xrightarrow{d} A$ and $B \xrightarrow{s} D \xrightarrow{t} B$ be splitting data for a and b . We set $g = sfd : C \rightarrow D$. One easily checks that $\mathring{f} \xrightarrow{(c,d)} \mathring{g} \xrightarrow{(d,t)} \mathring{f}$ is a splitting data of the idempotent $(a, b) : \mathring{f} \rightarrow \mathring{f}$. The converse statement is obvious. \square

7.5. Split idempotents in the category of pseudo-triangles. Let \mathcal{P} be a τ -category equipped with a pseudo-triangulated category structure given by a singular τ -extension

$$0 \rightarrow \Upsilon \xrightarrow{i} \text{Ptr} \xrightarrow{p} \mathcal{P}^{[1]} \rightarrow 0.$$

and τ -transformation $\varphi : \Delta \rightarrow \Upsilon$.

Lemma 7.5.1. *Let $x : [f] \rightarrow [f]$ be an idempotent in Ptr with $p(x) = (a, b) : \mathring{f} \rightarrow \mathring{f}$. Then x is a split idempotent in Ptr iff a and b are split idempotents in \mathcal{P} .*

Proof. If part is clear. Assume a and b are split idempotents. Then $(a, b) : \mathring{f} \rightarrow \mathring{f}$ is also split idempotent thanks to Lemma 7.4.1. Hence the result follows from Proposition 7.3.2. \square

As an immediate consequence of the above abstract non-sense we get the following crucial lemma in [12].

Corollary 7.5.2. *Let \mathcal{T} be a triangulated category and let*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u_f} & C_f & \xrightarrow{v_f} & A[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a[1] \\ A & \xrightarrow{f} & B & \xrightarrow{u_f} & C_f & \xrightarrow{v_f} & A[1] \end{array}$$

be a morphism of distinguished triangles. Assume a, b and c are idempotents. If a and b are split idempotents, then $(a, b, c) : [f] \rightarrow [g]$ is a split idempotent of the category Triangles_0 . In particular c is a split idempotent of \mathcal{T} .

7.6. The full embedding $\varrho : (\mathbb{A}^{\text{Ka}})^{[1]} \rightarrow (\mathbb{A}^{[1]})^{\text{Ka}}$. In this section we compare categories $(\mathbb{A}^{\text{Ka}})^{[1]}$ and $(\mathbb{A}^{[1]})^{\text{Ka}}$.

Let \mathbb{A} be an additive category. Objects of $(\mathbb{A}^{\text{Ka}})^{[1]}$ are arrows $f : (A, e) \rightarrow (A', e')$ in \mathbb{A}^{Ka} , where A and A' are objects of \mathbb{A} , while e and e' are idempotents of the category \mathbb{A} . We can also say that the objects of the category $(\mathbb{A}^{\text{Ka}})^{[1]}$ are diagrams in \mathbb{A}

$$\begin{array}{ccc} A & \xrightarrow{e} & A \\ f \downarrow & \searrow f & \downarrow f \\ A' & \xrightarrow{e'} & A' \end{array}$$

such that $e^2 = e, (e')^2 = e', fe = f = e'f$. Such an object is denoted by (A, e, A', e', f) .

On the other hand the objects of $(\mathbb{A}^{[1]})^{\text{Ka}}$ are pairs $(\overset{\circ}{f}, x)$, where $f : A \rightarrow B$ is an arrow in \mathbb{A} and $x = (e, e') : \overset{\circ}{f} \rightarrow \overset{\circ}{f}$ is an idempotent in $\mathbb{A}^{[1]}$. We can also say that the objects of the category $(\mathbb{A}^{[1]})^{\text{Ka}}$ are diagrams in \mathbb{A}

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ e \downarrow & & \downarrow e' \\ A & \xrightarrow{f} & A' \end{array}$$

with $e^2 = e, (e')^2 = e', fe = e'f$. Such an object is denoted by $[A, f, A', e, e']$. We see that the map

$$\varrho((A, e, A', e', f)) = [A, f, A', e, e']$$

yields an embedding of the class of objects of $(\mathbb{A}^{\text{Ka}})^{[1]}$ into the class of objects of $(\mathbb{A}^{[1]})^{\text{Ka}}$. The next lemma shows that ϱ can be extended as a full and faithful functor

$$\varrho : (\mathbb{A}^{\text{Ka}})^{[1]} \rightarrow (\mathbb{A}^{[1]})^{\text{Ka}}.$$

Lemma 7.6.1. *For any objects (A, e, A', e', f) and (B, d, B', d', g) of the category $(\mathbb{A}^{\text{Ka}})^{[1]}$ we have*

$$\text{Hom}_{(\mathbb{A}^{\text{Ka}})^{[1]}}((A, e, A', e', f), (B, d, B', d', g)) \cong \text{Hom}_{(\mathbb{A}^{[1]})^{\text{Ka}}}([A, f, A', e, e'], [B, g, B', d, d'])$$

Proof. A direct inspection shows that in both cases morphisms are pairs (h, h') , where $h : A \rightarrow B$ and $h' : A' \rightarrow B'$ are morphisms in \mathbb{A} such that

$$dh = h = hc, \quad d'h' = h' = h'c', \quad h'f = gh.$$

□

7.7. Idempotent completion of pseudo-triangulated categories. In this section we show that Karubization of a pseudo-triangulated category carries a natural pseudo-triangulated category structure (compare with [1]). This is based on the previous relationship between the categories $(\mathbb{A}^{\text{Ka}})^{[1]}$ and $(\mathbb{A}^{[1]})^{\text{Ka}}$.

Let \mathcal{P} be a τ -category equipped with a pseudo-triangulated category structure given by a singular τ -extension

$$0 \rightarrow \mathcal{Y} \xrightarrow{i} \text{Ptr} \xrightarrow{p} \mathcal{P}^{[1]} \rightarrow 0.$$

and τ -transformation $\varphi : \Delta \rightarrow \mathcal{Y}$. By passing to the idempotent completion we obtain another singular τ -extension (see lemma 7.3.1):

$$0 \rightarrow \mathcal{Y}^{\text{Ka}} \rightarrow \tilde{\text{Ptr}} \rightarrow (\mathcal{P}^{[1]})^{\text{Ka}} \rightarrow 0.$$

Now we can pull-back it along the ϱ to get a singular τ -extension

$$0 \rightarrow \mathcal{Y}^{\text{Ka}} \rightarrow \widehat{\text{Ptr}} \rightarrow (\mathcal{P}^{\text{Ka}})^{[1]} \rightarrow 0$$

which in fact is a pseudo-triangulated category structure on \mathcal{P}^{Ka} . One easily sees that for triangulated categories this is exactly the construction in [1].

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